Algebraic K-theory and traces for structured ring spectra

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Banff workshop, March 9–14, 2008

Outlines

Topological Hochschild homology

Topological cyclic homology

Algebraic K-theory of $\mathbb{S}$-algebras

The cyclotomic trace

$\mathbb{S}$-algebras

Recall that a symmetric spectrum A is a spectrum in which each space $A(n)$ is equipped with a based Σ_n-action such that the iterated structure maps

$$S^m \wedge A(n) \to A(m+n)$$

are $\Sigma_m \times \Sigma_n$-equivariant.

Let Sp^Σ be the symmetric monoidal category of symmetric spectra.

A symmetric ring spectrum is a monoid in Sp^Σ. Explicitly, have a unit $S^0 \to A(0)$ and multiplications $A(m) \wedge A(n) \to A(m+n)$.

Definition

In this talk an $\mathbb{S}$-algebra will mean a connective symmetric ring spectrum, that is, $\pi_i A = 0$ for $i < 0$.

(sometimes convenient to assume convergence.)

The cyclic bar construction

Let $(\mathcal{A}, \boxtimes, 1_{\mathcal{A}})$ be a symmetric monoidal category. For a monoid A in $\mathcal{A}$, the cyclic bar construction is the cyclic object

$$B^{\mathrm{cy}}_{\bullet}(A) \colon [k] \mapsto \underbrace{A \boxtimes \cdots \boxtimes A}_{k+1}$$

with face operators

$$d_i(a_0 \boxtimes \cdots \boxtimes a_k) = \begin{cases} a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_k, & i = 0, \ldots k-1, \\ a_k a_0 \boxtimes \cdots \boxtimes a_{k-1}, & i = k, \end{cases}$$

degeneracy operators

$$s_i(a_0 \boxtimes \cdots \boxtimes a_k) = a_0 \boxtimes \ldots a_i \boxtimes 1_{\mathcal{C}} \boxtimes a_{i+1} \cdots \boxtimes a_k, \quad i = 0, \ldots, k,$$

and cyclic operators

$$t_i(a_0 \boxtimes \cdots \boxtimes a_k) = a_k \boxtimes a_0 \boxtimes \cdots \boxtimes a_{k-1}.$$

Relation to the free loop space

Let G be a topological monoid with cyclic bar construction $B^{\mathrm{cy}}_{\bullet}(G)$ (originally introduced by Waldhausen).
The cyclic structure induces an action of the circle group $\mathbb{T}$ on the topological realization $B^{\mathrm{cy}}(G)$. Consider the composite map

$$\mathbb{T} \times B^{\mathrm{cy}}(G) \to B^{\mathrm{cy}}(G) \xrightarrow{p} B(G),$$

where p is the projection onto the usual bar construction

$$(a_0, \dots, a_k) \mapsto (a_1, \dots, a_k).$$

Theorem

If G is grouplike and well-based, then the adjoint map

$$B^{cy}(G) \to L(B(G)) = \mathrm{Map}(\mathbb{T}, B(G))$$

is a weak homotopy equivalence

Topological Hochschild homology

The topological Hochschild homology of an $\mathbb{S}$-algebra A may be defined as the derived tensor product

$$\mathrm{TH}(A) = A \wedge_{A^e} A,$$

where $A^e = A \wedge A^{\mathrm{op}}$ is the enveloping algebra.

Theorem (Shipley)

If A is a cofibrant $\mathbb{S}$-algebra, then $B^{\mathrm{cy}}(A)$ is a model of $\mathrm{TH}(A)$.

Here $B^{\mathrm{cy}}(A)$ is the realization of the cyclic bar construction in the symmetric monoidal category of symmetric spectra.

An analogous result was obtained earlier by Elmendorf-Kriz-May-Mandell in their setting.

Discrete rings

For a discrete ring A, let $\mathrm{TH}(A)$ be the topological Hochschild homology of the Eilenberg-Mac Lane spectrum HA.

In general, $\mathrm{TH}(A)$ is a generalized Eilenberg-Mac Lane spectrum for any discrete ring A.

Example (Bökstedt)

For the field $\mathbb{F}_p$, $\pi_* \mathrm{TH}(\mathbb{F}_p) \cong \mathbb{F}_p[x]$, where $|x| = 2$.

Theorem (Morita invariance)

For any discrete ring A,

$$\mathrm{TH}(M_n(A)) \simeq \mathrm{TH}(A).$$

In fact, $M_n(A)$ can be defined for any $\mathbb{S}$-algebra A (see later) and $\mathrm{TH}(A)$ satisfies Morita invariance in general.

Example: Spherical group rings

Let G be a topological monoid (for instance the Moore loops $\Omega(X)$ on a based space X). The spherical group ring of G is the $\mathbb{S}$-algebra defined by

$$\mathbb{S}[G] = \mathbb{S} \wedge G_+$$

Waldhausen observed that the k-simplices of $B^{\mathrm{cy}}_\bullet(\mathbb{S}[G])$ are given by

$$\underbrace{\mathbb{S}[G] \wedge \cdots \wedge \mathbb{S}[G]}_{k+1} = \mathbb{S} \wedge G_+^{k+1},$$

hence $\mathrm{TH}(\mathbb{S}[G]) \simeq \Sigma^\infty(B^{\mathrm{cy}}(G)_+)$. This implies

Theorem (Waldhausen)

For a connected based space X,

$$\mathrm{TH}(\mathbb{S}[\Omega(X)]) \simeq \Sigma^\infty(L(X)_+).$$

Example: Thom spectra

In general, one can associate a Thom $\mathbb{S}$-algebra $T(f)$ to any loop map $f\colon \Omega X \to BO$ (or BF).

In particular, the identity $BO \to BO$ and the realification map $BU \to BO$ give rise to the Thom spectra MO and MU.

Theorem (Blumberg-Cohen-Schlichtkrull)

If f is a 3-fold loop map, then $\mathrm{TH}(T(f)) \simeq T(f) \wedge X_+$.

For instance, $\mathrm{TH}(MU) \simeq MU \wedge SU_+$.

Remark

In fact, B-C-S gives an explicit description of $\mathrm{TH}(T(f))$ for any loop map f. For suitable chosen f this calculates $\mathrm{TH}(\mathbb{S}[\Omega X])$, $\mathrm{TH}(\mathbb{F}_p)$ and $\mathrm{TH}(\mathbb{Z}_{(p)})$.

Bökstedt's model for $\mathrm{TH}(A)$

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \dots, n\}$ and morphisms the injective maps.
An $\mathbb{S}$-algebra A gives rise to an $\mathcal{I}^{k+1}$-diagram for each k,

$$(\mathbf{n}_0, \dots, \mathbf{n}_k) \mapsto \mathrm{Map}_*(S^{n_0} \wedge \cdots \wedge S^{n_k}, A(n_0) \wedge \cdots \wedge A(n_k)).$$

Let $\mathrm{TH}_k(A)$ to be the symmetric spectrum with nth space

$$\operatorname*{hocolim}_{\mathcal{I}^{k+1}} \mathrm{Map}_*(S^{n_0} \wedge \cdots \wedge S^{n_k}, A(n_0) \wedge \cdots \wedge A(n_k) \wedge S^n).$$

The symmetric monoidal structure of $\mathcal{I}$ makes $[k] \mapsto \mathcal{I}^{k+1}$ a cyclic category, hence gives a cyclic spectrum $\mathrm{TH}_\bullet(A)$.

Bökstedt defines $\mathrm{TH}(A)$ as the topological realization.

Theorem (Shipley)

$\mathrm{TH}(A)$ *always represents the derived smash product* $A \wedge_{A^e} A$.

Advantages of Bökstedt's model

Bökstedt's model of $\mathrm{TH}(A)$ has the following convenient properties:

- it is homotopy invariant for all $\mathbb{S}$-algebras (at least if level-wise well based),
- the cyclic structure induces a $\mathbb{T}$-action on $\mathrm{TH}(A)$ making it a C_r-equivariant Ω-spectrum for all finite subgroups C_r,
- the fixed point spectra $\mathrm{TH}(A)^{C_r}$ have the "correct" homotopy types,
- it is easy to extend $\mathrm{TH}(A)$ to a genuine equivariant spectrum indexed on all $\mathbb{T}$-representations,
- it is easy to define the restriction maps used in the definition of topological cyclic homology.

Big Witt vectors

Let A be a commutative discrete ring. For each $n \geq 1$, let

$$\langle n \rangle = \{d \colon \ d \text{ divides } n\}.$$

The commutative ring $\mathbb{W}_{\langle n \rangle}(A)$ has underlying set $A^{\langle n \rangle}$. The ring structure is determined by the condition that the ghost coordinate map

$$\mathbb{W}_{\langle n \rangle}(A) \to A^{\langle n \rangle}, \quad \{\mathbf{a}_d\}_{d|n} \mapsto \{\mathbf{w}_d = \Sigma_{s|d} s \mathbf{a}_s^{d/s}\}_{d|n},$$

be a natural ring homomorphism to the product ring $A^{\langle n \rangle}$. There are *restriction homomorphisms*

$$R_r \colon \ \mathbb{W}_{\langle nr \rangle}(A) \to \mathbb{W}_{\langle n \rangle}(A),$$

such that $\mathbb{W}(A) = \lim_R \mathbb{W}_{\langle n \rangle}(A)$ has underlying set $A^{\mathbb{N}}$. The underlying abelian group has the description

$$\mathbb{W}(A) \simeq (1 + xA[[x]])^{\bullet}, \quad (\mathbf{a}_1, \mathbf{a}_2, \dots) \leftrightarrow \prod_{i=1}^{\infty}(1 - \mathbf{a}_i x^i).$$

The Frobenius homomorphisms

The Frobenius homomorphisms

$$F_r\colon\ \mathbb{W}_{\langle nr\rangle}(A) \to \mathbb{W}_{\langle n\rangle}(A)$$

are determined by the behavior on ghost coordinates

$$\{\mathbf{w}_d\}_{d|nr} \mapsto \{\mathbf{w}_{dr}\}_{d|n}.$$

Let $\mathbb{I}$ be the category with objects $1, 2, 3, \dots$ and two types of morphisms, $F_r, R_r\colon\ nr \to n$, subject to the relations

$$R_1 = F_1 = \mathrm{id}, \quad R_r R_s = R_{rs}, \quad F_r F_s = F_{rs}, \quad R_r F_s = F_s R_r.$$

The Witt vector construction gives a functor

$$\mathbb{W}_\bullet(A)\colon\ \mathbb{I} \to \text{Comm rings}, \qquad n \mapsto \mathbb{W}_{\langle n\rangle}(A).$$

The fixed points of $\mathrm{TH}(A)$

Theorem (Hesselholt-Madsen)

If A is a commutative $\mathbb{S}$-algebra, then $\pi_0 \mathrm{TH}(A)^{C_n} \cong \mathbb{W}_{\langle n \rangle}(\pi_0(A))$.

The restriction and Frobenius homomorphisms lift to maps of $\mathbb{S}$-algebras

$$F_r, R_r \colon \ \mathrm{TH}(A)^{C_{nr}} \to \mathrm{TH}(A)^{C_n}.$$

In fact, these are well-defined spectrum maps for **any** $\mathbb{S}$-algebra A and gives an $\mathbb{I}$-diagram of symmetric spectra $n \mapsto \mathrm{TH}(A)^{C_n}$.

Remark (Hesselholt)

Given a (not necessarily commutative) discrete ring A, one may **define**

$$\mathbb{W}_{\langle n \rangle}(A) = \pi_0 \mathrm{TH}(A)^{C_n}.$$

Topological cyclic homology

Definition (Bökstedt-Hsiang-Madsen)

The topological cyclic homology of an $\mathbb{S}$-algebra A is defined by

$$\mathrm{TC}(A) = \operatorname*{holim}_{\mathbb{I}} \mathrm{TH}(A)^{C_n}$$

The limit process may be broken up in two steps: let

$$\mathrm{TR}(A) = \operatorname*{holim}_{R} \mathrm{TH}(A)^{C_n},$$

then $\mathrm{TC}(A)$ is the homotopy fixed points,

$$\mathrm{TC}(A) = \mathrm{TR}(A)^{h\{F_r\}}$$

Notice, that $\pi_0 \mathrm{TR}(A) = \mathbb{W}(\pi_0(A))$

The p-typical version

For an $\mathbb{S}$-algebra A and a prime p, define

$$\mathrm{TR}(A,p) = \operatorname*{holim}_{R_p} \mathrm{TH}(A)^{C_{p^n}}.$$

If A is commutative, then $\pi_0 \, \mathrm{TR}(A,p) = W(\pi_0 A)$, the p-typical Witt vectors of $\pi_0 A$. Then $\mathrm{TC}(A,p)$ may be defined as the homotopy fibre

$$\mathrm{TC}(A,p) \to \mathrm{TR}(A,p) \xrightarrow{id-F} \mathrm{TR}(A,p).$$

Theorem (Goodwillie)

There is a canonical stable equivalence of p-completed spectra

$$\mathrm{TC}(A)^{\wedge}_{p} \xrightarrow{\simeq} \mathrm{TC}(A,p)^{\wedge}_{p}.$$

Example: $\mathrm{TC}(\mathbb{F}_p)$

Theorem (Hesselholt-Madsen)

There is a stable equivalence

$$\mathrm{TC}(\mathbb{F}_p) \simeq \mathrm{TC}(\mathbb{F}_p, p) \simeq H\mathbb{Z}_p^\wedge \vee \Sigma^{-1} H\mathbb{Z}_p^\wedge.$$

In fact, the cofibration sequence

$$\mathrm{TC}(\mathbb{F}_p, p) \to \mathrm{TR}(\mathbb{F}_p) \xrightarrow{id-F} \mathrm{TR}(\mathbb{F}_p)$$

reduces to

$$\mathrm{TC}(\mathbb{F}, p) \to H\mathbb{Z}_p^\wedge \xrightarrow{0} H\mathbb{Z}_p^\wedge.$$

Example: $\mathrm{TC}(\mathbb{S}[\Omega X])$

Theorem (Bökstedt-Hsiang-Madsen)

There is a homotopy cartesian square

$$\begin{tikzcd}
\mathrm{TC}(\mathbb{S}[\Omega X])^{\wedge}_{p} \arrow[r] \arrow[d] & \Sigma^{\infty}(\Sigma(L(X)_{h\mathbb{T}+}))^{\wedge}_{p} \arrow[d, "\mathrm{trf}_{\mathbb{T}}"] \\
\Sigma^{\infty}(L(X)_{+})^{\wedge}_{p} \arrow[r, "\mathrm{id}-\Delta_{p}"] & \Sigma^{\infty}(L(X)_{+})^{\wedge}_{p},
\end{tikzcd}$$

- $\Delta_p\colon L(X) \to L(X)$ *is induced by the p-power map on* $\mathbb{T}$,
- $\mathrm{trf}_{\mathbb{T}}$ *is the dimension shifting transfer.*

In particular, for $X = *$ this becomes

$$\mathrm{TC}(\mathbb{S})^{\wedge}_{p} \simeq \left(\Sigma^{\infty}(S^0) \vee \Sigma\mathbb{C}P^{\infty}_{-1}\right)^{\wedge}_{p}$$

Matrices and spaces of stable equivalences

For an $\mathbb{S}$-algebra A, let

$$M_n(A) = \mathrm{Hom}_A(A^{\vee n}, A^{\vee n}).$$

On homotopy groups: $\pi_* M_n(A) = M_n(\pi_* A)$.

The topological monoid

$$Q_{\mathcal{I}}(M_n(A)) = \operatorname*{hocolim}_{\mathbf{m} \in \mathcal{I}} \mathrm{Map}_*(S^m, M_n(A)(m))$$

represents the space of A-linear endomorphism of $A^{\vee n}$.

Definition

Let $GL_n(A)$ be the union of the components in $Q_{\mathcal{I}} M_n(A)$ that become invertible in

$$\pi_0 Q_{\mathcal{I}}(M_n(A)) = M_n(\pi_0 A).$$

Thus,

$$GL_n(A) \simeq \{\text{stable equivalences } A^{\vee n} \to A^{\vee n}\}.$$

Example

For a discrete ring A with Eilenberg-Mac Lane spectrum HA,

$$GL_n(HA) \simeq GL_n(A).$$

Definition

The units of an $\mathbb{S}$-algebra A is the topological monoid $GL_1(A)$.

Example

For the real and complex topological K-theory spectra ko and ku,

$$GL_1(ko) \simeq BO \times \{\pm 1\}$$

and

$$GL_1(ku) \simeq BU \times \{\pm 1\}.$$

Algebraic K-theory of $\mathbb{S}$-algebras

For an $\mathbb{S}$-algebra A, let $\mathcal{F}_A$ be the category of finitely generated free A-modules with objects $A^{\vee n}$.

The topological category $\omega\mathcal{F}_A$ of stable equivalences in $\mathcal{F}_A$ is defined by

$$\omega\mathcal{F}_A = \coprod_{n=0}^{\infty} GL_n(A).$$

(Suppressing the issue of invariant basis number.)

The symmetric monoidal structure of $\mathcal{F}_A$ gives a Γ-space with underlying space $\coprod_{n=0}^{\infty} BGL_n(A)$.

Definition

The algebraic K-theory spectrum $\mathrm{K}(A)$ is the spectrum associated to this Γ-space.

Thus,

$$\Omega^{\infty}\mathrm{K}(A) \simeq \text{“}\Omega B(\coprod_{n=0}^{\infty} BGL_n(A))\text{''}.$$

The trace map

Applying the cyclic bar construction to $\omega\mathcal{F}_A$ gives a Γ-space with underlying space

$$\coprod_{n=0}^{\infty} B^{\mathrm{cy}} GL_n(A).$$

Definition (Bökstedt-Hsiang-Madsen)

The cyclic algebraic K-theory spectrum $\mathrm{K}^{\mathrm{cy}}(A)$ is the spectrum associated to this Γ-space.

Thus,

$$\Omega^\infty \mathrm{K}^{\mathrm{cy}}(A) \simeq \text{``}\Omega B(\coprod_{n=0}^{\infty} B^{\mathrm{cy}} GL_n(A))\text{''}.$$

There is a natural map $\mathrm{K}(A) \to \mathrm{K}^{\mathrm{cy}}(A)$ induced (for instance) by

$$BGL_n(A) \to L(BGL_n(A)) \simeq B^{cy} GL_n(A),$$

Recall that in simplicial degree k,

$$\Omega^\infty \operatorname{TH}_k(A) = \operatorname*{hocolim}_{\mathcal{I}^{k+1}} \operatorname{Map}_*(\textstyle\bigwedge_{i=0}^{k} S^{n_i}, \bigwedge_{i=0}^{k} A(n_i)).$$

There is a canonical map

$$B^{\mathrm{cy}}(GL_n(A)) \to B^{\mathrm{cy}}(Q_{\mathcal{I}} M_n(A)) \to \Omega^\infty \operatorname{TH}(M_n(A))$$

induced in simplicial degree k by

$$\prod_{i=0}^{k} \operatorname{Map}_*(S^{n_i}, M_n(A)(n_i)) \to \operatorname{Map}_*(\textstyle\bigwedge_{i=0}^{k} S^{n_i}, \bigwedge_{i=0}^{k} M_n(A)(n_i)).$$

Using Morita invariance $\operatorname{TH}(M_n(A)) \simeq \operatorname{TH}(A)$, get a map

$$\mathrm{K}^{\mathrm{cy}}(A) \to \operatorname{TH}(A),$$

“counting the periodic orbits in $\mathrm{K}^{\mathrm{cy}}(A)$” (Goodwillie).

Definition (Waldhausen, Bökstedt-Hsiang-Madsen)

The trace map is defined by $\operatorname{tr}\colon \ \mathrm{K}(A) \to \mathrm{K}^{\mathrm{cy}}(A) \to \operatorname{TH}(A).$

Application: Waldhausen's splitting of $\mathrm{K}(\mathbb{S}[G])$

Let G be a grouplike topological monoid. There is a canonical monoid homomorphism

$$G \to GL_1(\mathbb{S}[G])$$

inducing a map of spectra

$$\Sigma^\infty(BG_+) \to \mathrm{K}(\mathbb{S}[G]).$$

Theorem (Waldhausen)

This is split injective in the stable category.

Proof: The retraction is defined by

$$\mathrm{K}(\mathbb{S}[G]) \xrightarrow{\mathrm{tr}} \mathrm{TH}(\mathbb{S}[G]) \xrightarrow{\sim} \Sigma^\infty(L(BG)_+) \xrightarrow{\mathrm{ev}} \Sigma^\infty(BG_+). \qquad \square$$

For X connected get splitting of $\Sigma^\infty(X_+) \to \mathrm{K}(\mathbb{S}[\Omega X])$.

The relation to manifold theory

Waldhausen's algebraic K-theory of spaces $A(X)$ can be identified with $\mathsf{K}(\mathbb{S}[\Omega X])$. Waldhausen defines the Whitehead spectrum $\mathsf{Wh}^{\mathrm{Diff}}(X)$ as the cofiber

$$\Sigma^\infty(X_+) \to \mathsf{K}(\mathbb{S}[\Omega X]) \to \mathsf{Wh}^{\mathrm{Diff}}(X).$$

Theorem (Waldhausen)

There is a canonical splitting

$$A(X) \simeq \Sigma^\infty(X_+) \vee \mathsf{Wh}^{\mathrm{Diff}}(X)$$

and if M is a compact smooth manifold, then

$$\Omega^2\Omega^\infty \mathsf{Wh}^{\mathrm{Diff}}(M) \simeq \mathcal{P}(M),$$

where $\mathcal{P}(M)$ is the stable pseudoisotopy space.

For instance,

$$\mathcal{P}(*) = \operatorname*{hocolim}_n \mathrm{Diff}(D^{n+1} \text{ rel } D^n).$$

The cyclotomic trace

The map $\mathsf{K}^{\mathrm{cy}}(A) \to \mathsf{TH}(A)$ lifts to maps

$$\mathsf{K}^{\mathrm{cy}}(A) \to \mathsf{TH}(A)^{C_n},$$

"counting orbits of period n".
These liftings are compatible with the restriction maps R_r and induce a map

$$\mathsf{K}^{\mathrm{cy}}(A) \to \mathsf{TR}(A) = \operatorname*{holim}_{R} \mathsf{TH}(A)^{C_n}.$$

The composition with $\mathsf{K}(A) \to \mathsf{K}^{\mathrm{cy}}(A)$ is fixed by the Frobenius homomorphisms F_r.

Definition (Bökstedt-Hsiang-Madsen)

The cyclotomic trace is the induced map

$$\mathrm{trc}\colon\ \mathsf{K}(A) \to \mathsf{TR}(A)^{h\{F_r\}} = \mathsf{TC}(A).$$

The relative theorem

Theorem (Dundas)

Let $A \to B$ be a map of $\mathbb{S}$-algebras such that $\pi_0 A \to \pi_0 B$ is surjective with nilpotent kernel. Then the diagram

$$\begin{tikzcd}
\mathsf{K}(A) \arrow[r, "\mathrm{trc}"] \arrow[d] & \mathsf{TC}(A) \arrow[d] \\
\mathsf{K}(B) \arrow[r, "\mathrm{trc}"] & \mathsf{TC}(B)
\end{tikzcd}$$

becomes homotopy cartesian after profinite completion.

The proof is based on the analogous theorem for simplicial rings due to McCarthy.

There also is an integral version due to Dundas-McCarthy.

These results were conjectured by Goodwillie.

Example: the linearization map

Applied to the linearization map $\mathbb{S}[\Omega X] \to H\mathbb{Z}[\pi_1 X]$, Dundas' theorem gives the homotopy cartesian diagram

$$\begin{tikzcd}
\mathrm{K}(\mathbb{S}[\Omega X])^\wedge_p \arrow[r, "\mathrm{trc}"] \arrow[d] & \mathrm{TC}(\mathbb{S}[\Omega X])^\wedge_p \arrow[d] \\
\mathrm{K}(\mathbb{Z}[\pi_1 X])^\wedge_p \arrow[r, "\mathrm{trc}"] & \mathrm{TC}(\mathbb{Z}[\pi_1 X])^\wedge_p
\end{tikzcd}$$

For $X = *$, this becomes

$$\begin{tikzcd}
\mathrm{K}(\mathbb{S})^\wedge_p \arrow[r, "\mathrm{trc}"] \arrow[d] & \mathrm{TC}(\mathbb{S})^\wedge_p \arrow[d] \\
\mathrm{K}(\mathbb{Z})^\wedge_p \arrow[r, "\mathrm{trc}"] & \mathrm{TC}(\mathbb{Z})^\wedge_p
\end{tikzcd}$$

Here the TC-terms have been calculated by Bökstedt-Hsiang-Madsen and Bökstedt-Madsen (see later).

Assuming the Quillen-Lichtenbaum conjecture, Rognes has analyzed the p-primary homotopy types of $\mathrm{K}(\mathbb{S})$ and $Wh(*)$ for regular primes.

Finite $W(k)$-algebras

Theorem (Hesselholt-Madsen)

If A is a finite $W(k)$-algebra over p-typical Witt vectors $W(k)$ over a perfect field k, then

$$\mathrm{K}(A)^\wedge_p \xrightarrow{\sim} \mathrm{TC}(A)^\wedge_p[0,\infty)$$

In particular, the theorem applies to finite $\mathbb{Z}^\wedge_p$-algebras.

Example (Bökstedt-Madsen)

For p odd there is a stable equivalence

$$\mathrm{K}(\mathbb{Z}^\wedge_p)^\wedge_p \simeq \mathrm{TC}(\mathbb{Z}^\wedge_p)^\wedge_p[0,\infty) \simeq j \vee \Sigma j \vee \Sigma^3 ku.$$

An analogous result for $p = 2$ has been obtained by Rognes.

Remark (Hesselholt-Madsen)

If A is a discrete ring which is finitely generated as a $\mathbb{Z}$-module, then $\mathrm{TC}(A) \xrightarrow{\sim} \mathrm{TC}(A^\wedge_p)$ after p-completion.

Applications to $\mathbb{S}$-algebras

If A is an $\mathbb{S}$-algebra such that $\pi_0 A$ is a finite $W(k)$-algebra, then

$$\mathsf{K}(A)^{\wedge}_{p} \xrightarrow{\simeq} \mathsf{TC}(A)^{\wedge}_{p}[0,\infty).$$

This follows from the homotopy cartesian square

$$\begin{tikzcd}
\mathsf{K}(A)^{\wedge}_{p} \arrow[r, "\mathrm{trc}"] \arrow[d] & \mathsf{TC}(A)^{\wedge}_{p} \arrow[d] \\
\mathsf{K}(\pi_0 A)^{\wedge}_{p} \arrow[r, "\mathrm{trc}"] & \mathsf{TC}(\pi_0 A)^{\wedge}_{p}.
\end{tikzcd}$$

This is used by Ausoni for $ku^{\wedge}_{p}$ and by Ausoni-Rognes for the Adam's summand l_p.