Master projects

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1 Sub-Riemannian structures from embedded CR-manifolds

Let $\pi : E \to M$ be a vector bundle. An almost complex structure on E is an endomorphism $J : E \to E$ such that $J^2 = -\operatorname{id}_E$. An almost CR-manifold is a real manifold M with an almost complex structure $J : H \to H$ on a subbundle $H \subseteq TM$. It is called a CR-manifold if J is *integrable*, i.e. if $H^{1,0} \subseteq H^{\mathbb{C}} = T^{1,0}M$ is the eigenspace of the eigenvalue i in the complexification $H^{\mathbb{C}}$ of H, then $[H^{1,0}, H^{1,0}] \subseteq H^{1,0}$.

One of the simplest CR-manifolds are obtained as follows. Let $\mathscr{N}$ be a complex manifold and define $M = \ker F$ for some real valued function $F : \mathscr{N} \to \mathbb{R}$. We assume that $\partial F \neq 0$ globally and define $H^{1,0} = \ker \partial F \subseteq TM^{\mathbb{C}}$, $\bar{H}^{1,0} = H^{0,1}$ and $H = \mathbb{R}(H^{1,0} \oplus H^{1,0})$. On such manifolds, the Levi-form is defined as

$$q(v, w) = \sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial \bar{z}_j \partial z_i} d\bar{z}_j \wedge dz_j(v, Jw), \qquad v, w \in H.$$

If q is positive definite, this induces and inner product on H. If so, (M, H, q) is an example of a CR-manifold and a sub-Riemannian manifold.

The master project can be given as follows. Given a complex manifold $\mathscr{N}$, what sub-Riemannian manifold can be obtained with the above construction? What are the intrinsic properties related determined by the map F?

References

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2 Isometries and operators on sub-Riemannian model spaces

Let (M, g) be a Riemannian manifold, i.e. a manifold where every there is a continuously varying inner product $g = \langle \cdot\,, \cdot \rangle_g$ on each tangent space T_xM, $x \in M$. An equivalence map for such objects is called an *isometry*, and is a diffeomorphism $\varphi : (M_1, g_1) \to (M_2, g_2)$ such that

$$\langle \varphi_* v, \varphi_* w \rangle_{g_2} = \langle v, w \rangle_{g_1}, \qquad v, w \in TM.$$

For any Riemannian manifold (M, g), the space of all isometries $\operatorname{Isom}(M, g)$ from (M, g) to itself forms a Lie group. (M, g) is called a homogeneous space if $G = \operatorname{Isom}(M, g)$ acts transitively. It is called a model space if for every linear isometry $q : T_xM \to T_yM$, $x, y \in M$, there is an isometry $\varphi \in G$ with $\varphi_{*,x} = q$. The only Riemannian model spaces are, up to scaling, the Euclidean space E^n, the spheres S^n and the hyperbolic space $\mathbb{H}^n$, and they are hence uniquely determined by their sectional curvature.

A sub-Riemannian manifold (M, H, g) is defined similarly to a Riemannian manifold, only that $g = \langle\,\cdot\,,\,\cdot\,\rangle_g$ is an inner product on a subbundle $H \subseteq TM$. We also assume that H satisfies a technical condition, called the bracket-generating condition. An isometry from the sub-Riemannian manifold (M, H, g) to itself is a smooth map $\varphi : M \to M$ such that $\varphi_* H \subseteq H$, and finally

$$\langle \varphi_* v, \varphi_* w\rangle_g = \langle v, w\rangle_g, \qquad v, w \in H.$$

All such isometries $\mathrm{Iso}(M, H, g)$ also form a Lie group in this case. A sub-Riemannian model space is similarly defined as a space such that for any linear isometry $q : H_x \to H_y$, $x, y \in M$, then there is an isometry satisfying $\varphi_*|_{H_x} = q$. There is a much larger class of such model spaces and they are still many unsolved questions related to them..

On sub-Riemannian model spaces, we have well defined operations to give us new model spaces

1. If (M, H, g) is a model space, then so is its nilpotentization $\mathrm{Nil}(M, H, g)$.
2. If (M, H, g) has non-zero holonomy, then its isometry group can be made into a sub-Riemannian model space.
3. If (M^1, H^1, g^1) and (M^2, H^2, g^2) are two model spaces, then we can define a rolling sum $M^1 \boxplus M^2$ of the two spaces, as long as the curvatures are different enough.

By combining these operations, we can make new constructions. For example, for $j = 1, 2$, let (N^j, E^j, h^j) be two flat sub-Riemannian model spaces, that is, spaces consisting of a nilpotent Lie group N^j with a left invariant sub-Riemannian structure (E^j, h^j). Then product $N^1 \times N^2$ is not a model space and their rolling sum $N^1 \boxplus N^2$ is not well defined as both spaces are flat. However, if there is any curved model space Σ with $\mathrm{Nil}(\Sigma) = N^1$, then we can define a kind of product N^3 of N^1 and N^2 by $N_3 = \mathrm{Nil}(\Sigma \boxplus N^2)$. One should be able to make such a construction in pure algebraic terms. We would like to understand how to define such products on stratified nilpotent Lie groups algebraically in this master project.

References

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3 Rolling in the upside-down

Consider the case of a ball rolling on the table. We assume that the surfaces are both rubber coated, with high friction. As a consequence, we have the limitations that we cannot slide the ball, nor twist it. Hence, we can only move or turn the ball by rolling it. This allows us to move in a maximal of two directions. However, our space of configurations is five dimensional, with two dimensions for the ball, two for the table and one coordinate for their relative rotation. We can thus consider this system as a sub-Riemannian manifold (M, H, g) with $\dim M$ and $\operatorname{rank} H = 1$. Such a system has an isometry group of maximal possible dimension.

We can define similarly sub-Riemannian manifolds with large isometry group by looking at two simply connected, complete, two-dimensional Riemannian manifolds of constant curvature rolling on each other without twisting or slipping. Recall that such constant curvature manifolds are the plane, spheres and different scalings of the hyperbolic plane.

The only restriction is that the curvature of the two manifolds involved in the rolling can not be equal. For example, if we are rolling two spheres against one another, then we cannot make a geometry on the five dimensional configuration space if the spheres have the same radius. However, if we define M_{r_1,r_2} as the result of rolling two spheres of respective radii $r_1 < r_2$, then by looking at the sub-Riemannian distance of M_{r_1,r_2} and taking the limit as $r_2 \to r_1$, we get a well defined metric space.

It is unclear how ro interpret these limit spaces geometrically. Furthermore, there turns out to be a whole class of sub-Riemannian spaces with a high number of symmetric ”on the other side” of these limit spaces. The objective of the master thesis will be to study these spaces beyond the limit spaces geometrically and perhaps as a dynamical system.

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4 Brownian motion and second order operators

Let L be a first order smooth differential operator on $\mathbb{R}^n$ without constant term, that is

$$L = \sum_{i=1}^{n} a_j \frac{\partial}{\partial x_j} \qquad a_j \in C^\infty(\mathbb{R}^n).$$

Write $u(t,x) = P_t f(x)$ for the solution of the equation $(\partial_t - L)u = 0$ and $u(0,x) = f(x)$ with f being a sufficiently nice function. All such solution are given by *the flow* of L. The flow is a function

$$(t,x) \mapsto \varphi_t(x) \in \mathbb{R}^n, \qquad x \in \mathbb{R}^n, t \in (-\varepsilon_-(x), \varepsilon_+(x)).$$

such that

$$\partial_t(f \circ \varphi_t) = (Lf) \circ \varphi_t.$$

Such a flow exists from the existence and uniqueness theorem of ODEs and the solution of the corresponding heat equation is given by $P_t f = f \circ \varphi_t$.

Consider next when $L = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_j^2}$ be (one half of) the Laplacian operator. Then we can still define a solution $u(t,x) = P_t f(x)$ of the heat equation $(\partial_t - L)u = 0$ with $u(0,x) = f(x)$. Then there is no direct analogue of a flow. However, there exists a probabilistic analogue. For any $x \in \mathbb{R}^n$, let $t \mapsto X_t(x)$ denote the Brownian motion starting at $x \in \mathbb{R}$. Roughly speaking , $t \mapsto X_t(x)$ start at x at time $t = 0$ and the position of time t is determined by the multivariate normal distribution with expectation x and standard deviation t. Then X_t plays the role of a flow is the sense that $P_t f = \mathbb{E}[f \circ X_t]$.

Similarly, if we have a second order operator $L = \sum_{i,j=1}^n a_{ij}\partial_i\partial_j + \sum_{k=1}^n b_k \partial_k$, $a_{ij}, b_k \in C^\infty(M)$ with (a_{ij}) only having non-negative eigenvalues, then we have a stochastic diffusion $(t,x) \mapsto X_t(x)$ such that $u(t,x) = P_t f(x) = \mathbb{E}[f(X_t(x))]$ is a solution[1] of the heat equation $(\partial_t - L)u$ with $u(0,x) = f(x)$.

One of the approaches to work with such stochastic diffusions, is to associate them with geometry. If $\det(a_{ij}) \neq 0$, we can introduce a Riemannian metric $g = \sum_{i,j=1}^n g_{ij} dx_i dx_j$ with $(g_{ij})^{-1} = (a_{ij})$.

[1] This statement is not completely true in general. It assumes a property called *stochastic completeness*. However, there is a more complicated statement that works in general.

If some of the eigenvalues of (a_{ij}) are zero, then we can still define a corresponding geometry called sub-Riemannian geometry.

The idea of the master project will be to study stochastic diffusions in Riemannian manifolds. Then we will focus on a problem related to the heat flow on a Riemannian or sub-Riemannian manifold and try to solve it using probabilistic methods.

References

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