MASTER PROJECTS

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1. Group of isometries of pseudo H-type algebras

A pseudo H-type algebra is a Lie algebra built from a representation of a Clifford algebra $\mathrm{Cl}_{r,s}$ generated by a pseudo-Euclidean space $\mathbb{R}^{r,s} = (\mathbb{R}^{r+s}, \langle . , . \rangle_{r,s})$:

$$J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(V).$$

We write $J_z \in \mathrm{End}(V)$ for a vector $z \in \mathbb{R}^{r,s}$. We assume that the vector space V admits a non-degenerate bilinear form $\langle . , . \rangle_V$ such that

$$\langle J_z u, v\rangle_V + \langle u, J_z v\rangle_V = 0 \quad \text{for any} \quad z \in \mathbb{R}^{r,s},\ u, v \in V.$$

The existence of such a bilinear form was proved in [1]. Note that if $s = 0$, then the form $\langle . , . \rangle_V$ is sign definite and if $s > 0$, the form $\langle . , . \rangle_V$ has the equal dimension of maximal positive and negative definite subspaces in V. The pseudo H-type algebra $\mathfrak{n}_{r,s}$ is a vector space $\mathbb{R}^{r,s} \oplus V$ endowed with the Lie bracket

$$\langle J_z u, v\rangle_V = \langle z, [u, v]\rangle_{r,s}.$$

The group of automorphisms $\mathrm{Aut}(\mathfrak{n}_{r,s})$ of the algebras $\mathfrak{n}_{r,s}$ were studied in [2, 3].

The main task of the master project is to determine the isometry group of $\mathfrak{n}_{r,s}$, i.e. the subgroup of $\mathrm{Aut}(\mathfrak{n}_{\mathrm{r,s}})$ such that is preserves the bilinear forms $\langle . , . \rangle_V$ and $\langle . , . \rangle_{r,s}$. The case $s = 0$ was described in [4].

References

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2. Tanaka prolongation of graded Lie algebras with sub-pseudo-Riemannian metric

Let us assume that we are given a graded Lie algebra

$$\mathfrak{n} = \mathfrak{g}_{-\mu} \oplus \mathfrak{g}_{-\mu+1} \oplus \ldots \oplus \mathfrak{g}_{-1}$$

such that

$$[\mathfrak{g}_{-1}, \mathfrak{g}_{-j}] = \mathfrak{g}_{-j-1}, \quad [\mathfrak{g}_{-1}, \mathfrak{g}_{-\mu}] = \{0\},$$

and a symmetric bilinear non-degenerate form $\langle . , . \rangle_{\mathfrak{g}_{-1}}\colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{R}$. Let us assume that G_0 is a subgroup of the group of automorphisms $\mathrm{Aut}(\mathfrak{n})$ such that G_0 preserves metric $\langle . , . \rangle_{\mathfrak{g}_{-1}}$. Let $\mathfrak{g}_0$ be a Lie algebra of G_0 and denote by $\mathfrak{g}$ the Lie algebra

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{g}_0, \quad [A, g] := Ag, \quad A \in g_0,\ g \in \mathfrak{n}.$$

It was shown in [1] that the Tanaka prolongation of $\mathfrak{g}$ is trivial if G_0 preserves a positive definite bilinear form.

The tasks of the master thesis:

- Study the construction of the Tanaka prolongation both theoretically [3] and by making use the software [2];
- Check the conjecture *the Tanaka prolongation of $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{g}_0$ is trivial for and G_0-invariant symmetric bilinear non-degenerate form on $\mathfrak{g}_{-1}$* on the software.

- Prove or disprove the conjecture theoretically. As a model example can be taken the pseudo H-type algebras with G_0 constructed in the first master project.

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3. Hölder properties of solutions to some PDE of divergent type

Consider a differential equation

$$\mathcal{A}(x, \nabla h) = 0 \tag{3.1}$$

for $x \in \Omega$, Ω is a domain in $\mathbb{R}^n$. Here we also assume

$\mathcal{A}1$ The map $\Omega \ni x \mapsto \mathcal{A}(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$; the map $\mathbb{R}^n \ni \xi \mapsto \mathcal{A}(x, \cdot)$ is continuous for almost all $x \in \Omega$.

Moreover the following conditions hold for all $\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$ and almost all $x \in \Omega$.

$\mathcal{A}2$ $\langle \mathcal{A}(x, \xi), \xi \rangle \geq |\xi|^n \theta_1(x)$;
$\mathcal{A}3$ $|\mathcal{A}(x, \xi)| \leq |\xi|^{n-1} \theta_2(x)$;
$\mathcal{A}4$ $\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle > 0$;
$\mathcal{A}5$ $\mathcal{A}(x, \lambda\xi) = \lambda|\lambda|^{n-2} \mathcal{A}(x, \xi)$ for all $\lambda \in \mathbb{R} \setminus 0$.

The week continuous solution to (3.1) is called *$\mathcal{A}$-harmonic function.* The properties of $\mathcal{A}$-harmonic functions for weights $\theta_1(x) = \theta_2(x)$ were studied in [1].

The tasks of the master thesis is to show that the $\mathcal{A}$-harmonic solution to (3.1) is Hölder continuous, following the outline of the proof of Theorem 6.6 [1].

Particularly, the solutions of equations of type (3.1) are quasiconformal homeomorphic and non-homeomorphic maps introduced and studied in [2, 3]. The appearance of different weight functions θ is important for the elasticity theory.

References

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4. Grassmannians on the complexified and quaternion Heisenberg groups

The Grassmann manifold in $\mathbb{R}^n$ is a set of k-dimensional subspaces and they proved to be widely used in mathematics, physics, as well in the engineering sciences [1].

The closest noncomutative analogue of the Euclidean space $\mathbb{R}^n$ are nilpotent groups, let say of step two. The analogous of the Grassmann manifold would be a collection of subgroups having equal topological and/or homogeneous dimensions. The first attempt to describe the Grassmann manifold on the Heisenberg group was done in [2], that was based on the serious of previous works, see for instance [3].

The tasks of the master thesis is to describe the Grassmann manifolds on the complex and quaternion versions of the Heisenberg group. To do it the student has to learn the structure of these groups and their homogeneous subgroups. The notions of topological and homogeneous dimensions plays an important role. The construction is very useful in the geometric measure theory, that is a hot topic in the last 15 years.

References

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