Theory and Applications of Hyperbolic Conservation Laws

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Project 1: Understanding Rankine-Hugoniot deficits

Singular solutions of hyperbolic conservation laws were pioneered by Korchinski[1] and discussed in detail in the early works of Colombeau and Oberguggenberger[2] and Keyfitz and Kranzer[3]. In particular, Keyfitz and Kranzer used the Rankine-Hugoniot deficit in the context of the system

$$u_t + (u^2 - v)_x = 0, \tag{1}$$

$$v_t + \left(\tfrac{1}{3}u^3 - u\right)_x = 0, \tag{2}$$

for which some Riemann problems cannot be solved because certain states in phase space can be shown to have compact Hugoniot locus. Thus, instead of requiring the usual Rankine-Hugoniot conditions, the equations

$$c[u] - [u^2 - v] = 0, \tag{3}$$

$$c[v] + \left[\tfrac{1}{3}u^3 - u\right] = \alpha(t), \tag{4}$$

are to be satisfied for discontinuous solutions. The function $\alpha(t)$ represents the Rankine-Hugoniot deficit, and the solutions found in this way were shown to be limits of a special viscous approximation.

It is now well understood that these solutions can be interpreted in terms of delta distributions using the weak asymptotic method. In clear terms, the deficit $\alpha(t)$ appears as the amplitude of a Dirac delta distribution in a solution of the form

$$u(x,t) = U(x,t),$$
$$v(x,t) = V(x,t) + \alpha(t)\delta(x - \sigma(t)),$$

where $U(x,t)$ and $V(x,t)$ are measurable functions of bounded variation (BV), $\sigma(t)$ is the location of the shock, and u and v are limits of smooth approximations. As shown by Danilov and Shelkovich[4], the relations (3) can be incorporated directly into the general framework of weak solutions, and the weak asymptotic limit can be used as an admissibility condition.

Recently, these notions have been extended to the complex-valued weak asymptotic method[5] which gives considerably more flexibility in finding solutions. For example, with the complex asymptotic method, it is possible to find admissible solutions for systems such as mentioned in[6] which could not previously be solved. Since the complex-valued weak asymptotic method introduces a large measure of flexibility, it is generally advisable to use classical notions of admissibility in addition to the weak asymptotic framework.

The physical meaning of the Dirac delta solutions of hyperbolic systems has been discussed extensibly in the literature. In fact, δ-distributions are experimentally confirmed as a solution of systems of conservation laws. This has recently been observed in nonlinear chromatography[7]. Namely, during an experiment involving chemical interactions of different substances, Mazzoti et. al. noticed abrupt

[1] C. Korchinski, PhD Thesis, Adelphi University, 1977

[2] J.-F. Colombeau and M. Oberguggenberger, Comm. Partial Differential Equations **15** (1990), 905–938

[3] B. Keyfitz and H.C. Kranzer, *J. Differential Equations* **118** (1995), 420–451

[4] V. Danilov and V.M. Shelkovich, *J. Differential Equations* **211** (2005), 333–281

[5] H. Kalisch and D. Mitrović, *Proc. Edinburgh Math. Soc.* **55** (2012), 711–729

[6] B. Hayes and P. G. LeFloch, Nonlinearity **9** (1996), 1547–1563

[7] M. Mazzotti, A. Tarafder, J. Cornel, F. Gritti and G. Guiochon, J. Chromatography A **1217** (2010), 2002–2012

increment of concentrations on specific isolated sets. The results are confirmed mathematically in many papers where the system of chromatography equations is considered, for example in[8].

Nevertheless, a nonzero $\alpha(t)$ in (3) may be interpreted as failure to satisfy some physical balance law, and the Rankine-Hugoniot deficit appears precisely because of some defect in the physical modeling leading to the system under study.

As an example, consider the shallow-water system which models the flow of an inviscid fluid in a long channel of small uniform width. The system is usually written in the form

$$\partial_t h + \partial_x (uh) = 0, \quad \text{(mass conservation)}, \tag{5}$$

$$\partial_t (uh) + \partial_x \left(u^2 h + g\frac{h^2}{2} \right) = 0, \quad \text{(momentum conservation)}. \tag{6}$$

For smooth solutions, an equivalent system is

$$\partial_t h + \partial_x (uh) = 0, \quad \text{(mass conservation)}, \tag{7}$$

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + gh \right) = 0, \quad \text{(conservation of energy per unit mass)}, \tag{8}$$

and it can be shown that mass and momentum conservation in discontinuous solutions lead to a Rankine-Hugoniot deficit in (8) which takes the form of a loss of energy per unit mass with $\alpha'(t) = g\mathcal{H}$ for a certain constant $\mathcal{H}$. On the other hand, in some cases it is more reasonable to require conservation of particle energy (8) or total energy through the shock, and in this case a Rankine-Hugoniot deficit will be introduced in (6).

Our goal in this project is the incorporation of some of the above mentioned concepts into the existence / admissibility theory and the numerical approximation of hyperbolic conservation laws, and mapping Riemann problems between systems related by nonlinear transformation of the unknown variables, such as detailed in[9]. Another possible focus point is the attempt to define appropriate admissibility conditions to provide uniqueness for the solutions of conservation laws. The idea for introducing an admissibility concept is to consider a system of conservation laws as set of equations in which one of them is satisfied in the standard weak sense, while the others serve to correct the inadequacy of the model [10].

[8] B. Keyfitz, C. Tsikkou, Quart. Appl. Math. **70** (2012), 407–436

[9] H. Kalisch, D. Mitrovic, V. Teyekpiti, Nonlinearity **31** (2018), 5463

[10] H. Kalisch and D. Mitrovic, Int. J. Appl Comp Math. **8** (2022), 175

References

[1] C. Korchinski, *Solution of a Riemann problem for a* 2×2 *system of conservation laws possessing no classical weak solution*, PhD Thesis, Adelphi University, 1977.

[2] J.-F. Colombeau and M. Oberguggenberger, *On a hyperbolic system with a compatible quadratic term: generalized solutions, delta waves, and multiplication of distributions*, Comm. Partial Differential Equations **15** (1990), 905–938.

[3] B. Keyfitz and H.C. Kranzer, *Spaces of weighted measures for conservation laws with singular shock solutions* J. Differential Equations **118** (1995), 420–451.

[4] V.G. Danilov and V.M. Shelkovich, *Dynamics of propagation and interaction of* δ*-shock waves in conservation law system*, J. Differential Equations **211** (2005), 333–381.

[5] H. Kalisch and D. Mitrović, *Singular solutions of a fully nonlinear* $2x2$ *system of conservation laws*, Proc. Edinb. Math. Soc. **55** (2012), 711–729.

[6] B. Hayes and P. G. LeFloch, *Measure-solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity **9** (1996), 1547–1563.

[7] M. Mazzotti, A. Tarafder, J. Cornel, F. Gritti and G. Guiochon, *Experimental evidence of a* δ*-shock in nonlinear chromatography*, J. Chromatography. A, **1217(13)** (2010), 2002–2012.

[8] B. Keyfitz, and C. Tsikkou, *Conserving the wrong variables in gas dynamics: a Riemann solution with singular shocks*, Quart. Appl. Math. **70** (2012), 407–436.

[9] H. Kalisch, D. Mitrović and V.Teyekpiti, *Existence and uniqueness of singular solutions for a conservation law arising in magnetohydrodynamics*, Nonlinearity **31** (2018), 5463.

[10] H. Kalisch and D. Mitrovic, *On Existence and Admissibility of Singular Solutions for Systems of Conservation Laws*, International Journal of Applied and Computational Mathematics **8** (2022), p.175.