Master's thesis project: Existence of solutions to nonlinear wave equations

Sigmund Selberg

The project deals with the initial value problem for nonlinear wave equations

$$u_{tt} - \Delta u = F(u_t, \nabla_x u, \nabla_x^2 u), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d. \tag{1}$$

Here the unknown is a function $u = u(t, x)$, and F is a given smooth function vanishing to at least second order at the origin (meaning that F and its first order partial derivatives vanish there). This type of equation arises in a variety of physical problems involving wave propagation. Given initial data at time $t = 0$,

$$u(0, x) = \varepsilon f(x), \qquad u_t(0, x) = \varepsilon g(x), \tag{2}$$

where $\varepsilon > 0$ is a parameter, and f and g are assumed to be smooth and compactly supported on $\mathbb{R}^d$, the following questions are considered:

(i) (Local existence.) Does there exist a smooth solution $u(t, x)$ to (1), (2) on $[0, T] \times \mathbb{R}^d$ for some $T > 0$? Is the solution unique?

(ii) (Long-time/global existence for small data.) If the answer to the first question is yes, then what can be said about the life span T_ε of the smooth solution as ε tends to zero? By the *life span* we mean the supremum of $T > 0$ such that (1), (2) has a smooth solution on $[0, T) \times \mathbb{R}^d$. If $T_\varepsilon = +\infty$, we say that the solution exists globally.

(iii) Moreover, one may investigate to what extent the answer to the previous question depends on the structure of the function F and the space dimension d.

The local existence (i) is classical. An exposition can be found, for example, in the monographs [1, 6]. The typical way to prove local existence is by Picard iteration, making use of energy methods and Sobolev inequalities, and the first step will be to familiarise oneself with the techniques involved. The small data problem (ii), (iii) was considered in [2, 3, 4, 5], and is discussed also in [1, 6]. The intuition is that as ε tends to zero, the solution should in some sense approach the zero solution, which of course extends globally, and therefore one expects that T_ε tends to infinity.

It is a good idea to start with the semilinear case, where $F = F(u_t, \nabla_x u)$. If F vanishes to p-th order at the origin ($p \geq 2$), then by a Picard iteration one can show $T_\varepsilon \geq \frac{c}{\varepsilon^{p-1}}$, where $c > 0$ is a constant independent of ε, just by using (i) the energy inequality

$$\|\partial u(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq \varepsilon \left(\|\nabla f\|_{L^2(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)} \right) + \int_0^t \|G(s, \cdot)\|_{L^2(\mathbb{R}^d)} \, ds \tag{3}$$

for the linear wave equation $u_{tt} - \Delta u = G(t, x)$ with data (2), and (ii) the Sobolev inequality

$$|f(x)| \leq C \sum_{|\alpha| \leq N} \|\nabla^\alpha f\|_{L^2(\mathbb{R}^d)} \qquad (N > d/2). \tag{4}$$

The next step is to try to improve the lower bound $T_\varepsilon \geq \frac{c}{\varepsilon^{p-1}}$. This can be achieved by making use of dispersive properties of the free wave equation

$$u_{tt} - \Delta u = 0.$$

In fact, due to waves spreading out in different directions, a solution to this equation with smooth and compactly supported initial data has the decay

$$|u(t,x)| \leq \frac{C}{(1+|t|)^{\frac{d-1}{2}}},$$

and it is not unreasonable to expect that the solution to (1), (2) has the same property when $\varepsilon > 0$ is sufficiently small. This is indeed the case, as shown by Klainerman [4], who proved a replacement for the Sobolev inequality (4), for a function $u(t,x)$ on space-time, capturing the above decay property. This involves extending the Sobolev norm on the right hand side of (4) to a larger sum over a Lie algebra of vector fields generating the full Poincaré group plus the space-time dilation, instead of just using the translational vector fields ∂_{x_j}. These vector fields have favourable commutation properties relative to the wave operator $\partial_t^2 - \Delta$, allowing to use the energy method in combination with the generalized Sobolev inequality.

References

1. Lars Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 26, Springer-Verlag, Berlin, 1997. MR 1466700
2. Fritz John, *Delayed singularity formation for solutions of nonlinear partial differential equations in higher dimensions*, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), no. 2, 281–282. MR 390518
3. Sergiu Klainerman, *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math. **33** (1980), no. 1, 43–101. MR 544044
4. ———, *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math. **38** (1985), no. 3, 321–332. MR 784477
5. Jalal Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Differential Equations **46** (1982), no. 3, 409–425. MR 681231
6. Christopher D. Sogge, *Lectures on non-linear wave equations*, second ed., International Press, Boston, MA, 2008. MR 2455195