From Tree-Depth to Shrub-Depth, Evaluating MSO-Properties in Constant Parallel Time

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Theorem (Courcelle, 1990)

Every problem definable in monadic second-order logic (MSO) *can be decided in linear time on graphs of bounded tree-width.*

In particular, the 3-colorability problem can be solved in linear time on graphs of bounded tree-width.

Model-checking monadic second-order logic (MSO) *on graphs of bounded tree-width is fixed-parameter tractable.*

MSO is the restriction of second-order logic in which every second-order variable is a set variable.

A graph G is 3-colorable if and only if

$$G \models \exists X_1 \exists X_2 \exists X_3 \left(\forall u \bigvee_{1 \leqslant i \leqslant 3} X_i u \wedge \forall u \bigwedge_{1 \leqslant i < j \leqslant 3} \neg (X_i u \wedge X_j u) \right.$$
$$\left. \wedge \forall u \forall v \left(Euv \to \bigwedge_{1 \leqslant i \leqslant 3} \neg (X_i u \wedge X_i v) \right) \right).$$

MSO can also characterize Sat, Connectivity, Independent-Set, Dominating-Set, etc.

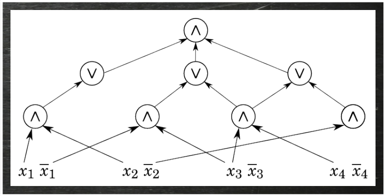
Constant Parallel Time = AC^0-Circuits.

A family of Boolean circuits $(C_n)_{n\in\mathbb{N}}$ are AC^0-circuits if for every $n \in \mathbb{N}$

(i) C_n computes a Boolean function from $\{0,1\}^n$ to $\{0,1\}$;

(ii) the depth of C_n is bounded by a fixed constant;

(iii) the size of C_n is polynomially bounded in n.



AC^0 and parallel computation

AC^0-circuits	parallel computation
# of input gates	length of input
depth	# of parallel computation steps
size	# of parallel processes

Theorem (Barrington, Immerman, and Straubing, 1990)
A problem can be decided by a family of dlogtime uniform AC^0-circuits if and only if it is definable in first-order logic (FO) with arithmetic.

For our purpose, it suffices to know the easy direction that every problem definable in FO can be computed by AC^0-circuits.

Using [Beame, Impagliazzo, and Pitass, 1998]:

Theorem

There is a class K *of graphs of bounded tree-width (even bounded path-width) and an* MSO*-formula* φ *such that model checking* φ *on* K *cannot be solved by* AC^0*-circuits.*

Theorem (C. and Flum, 2018)

On graphs of tree-depth $\leqslant \mathrm{d}$ *every problem definable in* MSO *can be decided by* AC^0*-circuits of depth* $\mathrm{O}(\mathrm{d})$.

Model-checking MSO *on graphs of bounded tree-depth is in* para-AC^0.

Theorem (C. and Flum, 2019)

On graphs of bounded shrub-depth every problem definable in MSO *can be decided by* AC^0*-circuits.*

Model-checking MSO *on graphs of bounded shrub-depth is in* para-AC^0.

para-AC^0 is an analog of AC^0 in the parameterized world.

Definition (Bannach, Stockhusen, and Tantau, 2015)

A parameterized problem Q is in para-AC^0 if there exists a family $(C_{n,k})_{n,k \in \mathbb{N}}$ of circuits such that:

1. The depth of every $C_{n,k}$ is bounded by a fixed constant.
2. $|C_{n,k}| \leqslant f(k) \cdot n^{O(1)}$ for every $n, k \in \mathbb{N}$.
3. Let $(x, k) \in \Sigma^* \times \mathbb{N}$. Then $\big((x, k) \in Q$ if and only if $C_{|x|,k}(x) = 1\big)$.
4. There is a TM that on input $(1^n, 1^k)$ computes the circuit $C_{n,k}$ in time $g(k) + O(\log n)$.

Both $f, g : \mathbb{N} \to \mathbb{N}$ are computable functions.

Theorem (Bannach, Stockhusen, and Tantau, 2015)

The parameterized vertex cover problem is in para-AC^0.

Theorem (C. , Flum, and Huang, 2017)

For every fixed $k \in \mathbb{N}$ *the k-vertex-cover problem can be solved by* AC^0*-circuits of depth at most* 34.

The tree-depth of a graph G, introduced by J. Nešetřil and P. Ossona de Mendez in 2006, measures how G is similar to a star.

The following are equivalent for a class of graphs K.

1. K has bounded tree-depth.
2. Every graph in K has a tree-decomposition of bounded width and bounded depth.
3. There is a constant $d \in \mathbb{N}$ such that every graph in K contains no path of length $\geqslant d$.

Shrub-depth

The shrub-depth of a graph class, introduced by R. Ganian, P. Hliněný, J. Nesetril, J. Obdrzálek, P. Ossona de Mendez, and R. Ramaduraiin in 2012, as an extension of tree-depth to dense graphs, similarly as clique-width is an extension of tree-width to dense graphs.

Shrub

Shrubs are perennial woody plants, and therefore have persistent woody stems above ground (compare with herbaceous plants). Usually shrubs are distinguished from trees by their height and multiple stems.

– Wikipedia

dense shrub

sparse tree

A graph $G = (V(G), E(G))$ is sparse if the number $|E(G)|$ of edges in G is linear in terms of the number $|V(G)|$ of vertices in G.

1. Trees.
2. Graphs of bounded tree-width.
3. Planar graphs.
4. Graphs excluding a fixed minor.
5. Graphs of bounded expansion.

Example

Let G be a sparse graph. Then its complement

$$G^c = \left(V(G), \binom{V(G)}{2} \setminus E(G)\right)$$

is dense.

The edges of G^c can be defined by FO-formula

$$\mathit{edge}(x, y) = x \neq y \wedge \neg Exy.$$

So one natural strategy is to obtain dense graphs by FO-interpretation of sparse graphs, which is computable by AC^0-circuits.

$$G^c \models \exists x \exists y Exy \iff G \models \exists x \exists y (x \neq y \wedge \neg Exy).$$

Let K ba a sparse graph class and I an FO-interpretation.

$$\begin{array}{ccc} G \in \mathsf{K} & \xrightarrow{I} & I(G) \in I(\mathsf{K}) \\ \downarrow & & \downarrow \\ G \models \varphi^I & \xrightarrow{I} & I(G) \models \varphi \end{array} \quad .$$

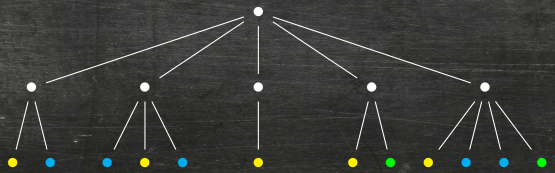
1. First we define the class $\textsc{Tree}[m, d]$ of trees of depth d with m labels.
2. Using a signature D any $T \in \textsc{Tree}[m, d]$ can be transferred to a shrub G.

TREE$[m, d]$ is the class of rooted trees with m labels and of depth d:

1. every root-to-leaf path has length d,
2. every leaf t is labelled with a color in $c(t) \in [m]$.

Example

A rooted tree in TREE$[3, 2]$.



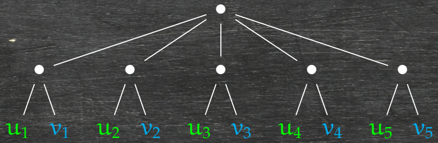
Definition

A tree-model of m labels and depth d of a graph G is a pair (T, D) of a rooted tree $T \in \textsc{Tree}[m, d]$ and a signature

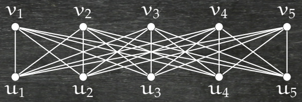
$$D \subseteq \{1, 2, \ldots, m\}^2 \times \{2, 4, \ldots, 2 \cdot d\}$$

for some $h \geqslant d$ such that

1. for any $i, j \in [m]$ and $s \in [d]$ if $(i, j, s) \in D$, then $(j, i, s) \in D$,
2. $V(G) = \mathit{leaves}(T)$,
3. $E(G) = \Big\{\{u, v\} \;\Big|\; u, v \in V(G) \text{ and } (c(u), c(v), \mathit{dist}^T(u, v)) \in D\Big\}$.

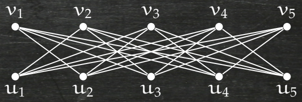


$$D := \{({\color{green}\bullet}, {\color{cyan}\bullet}, 2), ({\color{green}\bullet}, {\color{cyan}\bullet}, 4)\}$$



complete bipartite graph $K_{5,5}$

$$D := \{({\color{green}\bullet}, {\color{cyan}\bullet}, 4)\}$$



almost complete bipartite graph $B_{5,5}$

Definition
$TM_m(d)$ is the class of graphs with a tree-model in $\textsc{Tree}[m, d]$.

Definition
Let K be a class of graphs. Then K has shrub-depth d, if

1. $K \subseteq TM_m(d)$ for for some $m \in \mathbb{N}$,
2. $K \not\subseteq TM_{m'}(d-1)$ for for every $m' \in \mathbb{N}$.

Examples

1. The class of complete graphs has shrub-depth 1.
2. The class of complete bipartite graphs $K_{n,n}$ has shrub-depth 1.
3. The class of almost complete bipartite graphs $B_{n,n}$ has shrub-depth 2.
4. The class of trees of depth d has shrub-depth d.

Theorem

Let K *be class of graphs of bounded shrub-depth. Then model-checking* MSO *on* K *can be computed by parameterized* AC^0*-circuits.*

In particular, the 3-colorability problem can be solved in constant parallel time on graphs of bounded shrub-depth.

$$\begin{array}{ccc} G \in K & \xrightarrow{\mathrm{I}} & \mathrm{I}(G) \in \mathrm{I}(K) \\ \downarrow & & \downarrow \\ G \models \varphi^{\mathrm{I}} & \xrightarrow{\mathrm{I}} & \mathrm{I}(G) \models \varphi \end{array}.$$

The difficulty is to compute

$$\mathrm{I}(G) \mapsto G.$$

The square graph G^2 of G has $V(G^2) := V(G)$ and

$$E(G^2) := E(G) \cup \{\{u, v\} \mid \{u, w\}, \{w, v\} \in E(G) \text{ for some } w\}$$

Theorem (Motwani and Sudan, 1994)
$G^2 \mapsto G$ *is hard for* NP.

Our technical road map

1. The boundedness of shrub-depth is equivalent to the boundedness of SC-depth, i.e., subset-complementation depth.
2. Logic definability of SC-depth and SC-derivation.
3. From any SC-derivation we can compute a tree-model by AC^0-circuits.
4. On the tree-model, we can evaluate any MSO-formula φ by AC^0-circuits.

SC-Depth

Definition

Let $G = (V(G), E(G))$ be a graph and $S \subseteq V(G)$. We obtain G^S by flipping the edges within S. More precisely,

$$\begin{aligned} V(G^S) &:= V(G) \\ E(G^S) &:= \big\{\{u,v\} \in E(G) \;\big|\; u \notin S \text{ or } v \notin S\big\} \\ &\quad \cup \big\{\{u,v\} \notin E(G) \;\big|\; u \neq v \text{ and } u,v \in S\big\}. \end{aligned}$$

For every $i \in [n]$ let G_i be a graph consisting of the edge $\{a_i, b_i\}$.

1. $S_a := \{a_1, \ldots, a_n\}$.
2. $S_b := \{b_1, \ldots, b_n\}$.
3. $S := \{a_1, b_1, \ldots, a_n, b_n\}$.

Then

$$\left(\left(\left(G_1 \dot\cup G_2 \dot\cup \cdots \dot\cup G_n\right)^{S_a}\right)^{S_b}\right)^{S}$$

is the almost complete bipartite graph $B_{n,n}$.

Definition

1. $\mathrm{SC}(0)$ is the class of graphs whose vertex set is a singleton.
2. Assume $G_1, \ldots, G_m \in \mathrm{SC}(d)$ with pairwise disjoint vertex sets, and $S \subseteq V(G_1) \cup \cdots \cup V(G_m)$. Then
$$G \in \left(G_1 \dot\cup \cdots \dot\cup G_m\right)^S \in \mathrm{SC}(d+1).$$

Definition

The SC-depth $\mathrm{SC}(G)$ of G is the minimum d with
$$G \in \mathrm{SC}(d).$$

Theorem (Ganian et. al, 2012)

1. $\mathrm{TM}_m(d) \subseteq \mathrm{SC}\big(d \cdot m \cdot (m+1)\big)$.
2. $\mathrm{SC}(d) \subseteq \mathrm{TM}_{2^d}(d)$.

Corollary

$\mathcal{K}$ *has bounded shrub-depth if and only if it has bounded* SC*-depth.*

The Logic Definability of SC-Depth

Recall:

Definition

1. $\mathrm{SC}(0)$ is the class of graphs whose vertex set is a singleton.
2. Assume $G_1, \ldots, G_m \in \mathrm{SC}(d)$ with pairwise disjoint vertex sets, and $S \subseteq V(G_1) \cup \cdots \cup V(G_m)$. Then
$$G \in \left(G_1 \,\dot\cup\, \cdots \,\dot\cup\, G_m\right)^S \in \mathrm{SC}(d+1).$$

Theorem (folklore)

$\mathrm{SC}(d)$ *is definable in* MSO. *That is, there is an* MSO*-formula* φ_d *such that for every graph* G

$$G \models \varphi_d \quad \Longleftrightarrow \quad G \in \mathrm{SC}(d).$$

We have developed some sophisticated combinatorial machinery tailored for FO.

Theorem (FO approximation)

Let $d \geqslant 1$. *Then, there is a constant* $h \geqslant d$ *and an* FO-*formula* φ_d *such that for every graph* G

1. *if* $G \in SC(d)$, *then* $G \models \varphi_d$;
2. *if* $G \models \varphi_d$, *then* $SC(G) \leqslant h$.

Remark

The complexity of φ_d *is astronomical. The quantifier rank of* φ_3 *is at least*

$$2^{257} \geqslant 128 \times 10^{75}.$$

Thus φ_3 *cannot be written down within the known universe.*

Recall:

Theorem
$\mathrm{SC}(d) \subseteq \mathrm{TM}_{2^d}(d)$, *i.e., every graph* $G \in \mathrm{SC}(d)$ *has a tree-model in* $\textsc{Tree}[2^d, d]$.

Theorem
Let $d \geqslant 1$. *Then, there is a constant* $h \geqslant d$ *and an* FO*-interpretation* I *such that for any ordered graph* $(G, <)$ *with* $\mathrm{SC}(G) \leqslant d$

$$I(G, <) \text{ is a tree-model of } G \text{ in } \textsc{Tree}[2^h, h].$$

In other words, a tree model of G *can be computed by* AC^0*-circuits.*

1. Any class of graphs K of bounded shrub-depth is contained in some $SC(d)$.
2. There is an $h \geqslant d$ and an FO-interpretation I_1 such that for every $G \in SC(d)$ we have

$$T := I_1(G, <) \in \textsc{Tree}[2^h, h]$$

3. There is an FO-interpretation I_2 such that

$$I_2(T) = G.$$

Theorem (C. and Flum, 2018)

Model-checking MSO *for* $\textsc{Tree}[m, d]$ *can be computed by parameterized* AC^0*-circuits.*

Model-checking MSO for graphs of bounded shrub-depth

Let K be a class of graphs of bounded shrub-depth.

1. $G \in K \subseteq SC(d)$.
2. $T := I_1(G, <) \in \textsc{Tree}[2^h, h]$, where $<$ is an arbitrary ordering. Thus, T can be computed by AC^0-circuits,
3. $I_2(T) = G$. Thus, for any MSO-sentence φ
$$G \models \varphi \iff T \models \varphi^{I_2}.$$
4. Checking $T \models \varphi^{I_2}$ can be done by parameterized AC^0-circuits, given T and φ^{I_2}.
5. Checking $G \models \varphi$, i.e., $I_1(G, <) \models \varphi^{I_2}$ can be done by parameterized AC^0-circuits, given G and φ.

Model-checking MSO for graphs of bounded shrub-depth

Let K be a class of graphs of bounded shrub-depth.

Theorem

Model-checking MSO *on* K *can be computed by parameterized* AC^0 *circuits.*

1. Model-checking MSO on graphs of bounded shrub-depth (hence also tree-depth) is in para-AC^0.
2. The main technical tool is a combinatorial characterization of SC-depth which is definable in FO, with enormous hidden constants. Can we do better?
3. The next step is to extend our result to graphs of structurally bounded expansion.

Thank You!