# Diverse Fixed-Parameter Tractability

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# The Diverse X Paradigm

## Example: Diverse Vertex Cover

- Input: Graph $G$, integers $k, r, d$
- Parameter: $(k, r)$
- Question: Does $G$ have a list $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ of vertex covers:
  - Each $S_i$ is of size at most $k$, and
  - Some specified diversity measure of the set  is at least $d$
    - E.g: The sum of pairwise Hamming distances of the $S_i$s is at least $d$
    - The Hamming distance of sets $X, Y$ is $|X \setminus Y| + |Y \setminus X|$

# The Diverse X Paradigm

## Motivation

- Combinatorial formulations ignore lots of *side information*
  - Domain knowledge, trade secrets, ...
    - Floor plans, Galle (1989)
    - Scheduling, Weihe (via Mike)
  - Finding one best solution may not help
  - Given many good quality solutions, an expert can choose
- Finding *all* solutions of good quality may be too much
  - Find a sufficiently *diverse* set of good quality solutions
- Bootstrapping heuristics

# Diverse Vertex Cover

## Diversity Measure: Sum of Pairwise Hamming Distances

- Input: Graph $G$, integers $k, r, d$
- Parameter: $(k, r)$
- Question: Does $G$ have a list $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ of vertex covers:
  - Each $S_i$ is of size at most $k$, and
  - The sum of pairwise Hamming distances of the $S_i$s is at least $d$
    - The Hamming distance of sets $X, Y$ is $|X \setminus Y| + |Y \setminus X|$

### Theorem

*Diverse Vertex Cover is FPT.*

# A simple FPT Algorithm for Diverse Vertex Cover

## Diversity Measure: Sum of Pairwise Hamming Distances

- Observations:
  1. Every vertex cover in the list $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ contains a *minimal* vertex cover of size at most $k$.

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  2. All such minimal vertex covers can be enumerated in time $\mathcal{O}^\star(2^k)$
     - The leaves of the simple branching tree.

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- Observations:
  1. Every vertex cover in the list $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ contains a *minimal* vertex cover of size at most $k$.
  2. All such minimal vertex covers can be enumerated in time $\mathcal{O}^\star(2^k)$
     - The leaves of the simple branching tree.
  3. The diversity $d$ is never more than $k \cdot r^2$
     - $\binom{r}{2}$ pairs of vertex covers
     - Each pair has diversity at most $2k$

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   - Let this set be MinVC
3. Guess $S'_1 \subseteq S_1, S'_2 \subseteq S_2, \ldots, S'_r \subseteq S_r$ from MinVC
   - Each $S'_i$ is an element of MinVC
   - Possibly with repetition
   - The list $\{S'_1, S'_2, \ldots, S'_r\}$ satisfies two of the three requirements for a solution
     - TODO: Make the sum of pairwise Hamming distances to be at least $d$

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   - If this is at least $d$, stop and return this list

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3. Guess $S_1' \subseteq S_1, S_2' \subseteq S_2, \ldots, S_r' \subseteq S_r$ from MinVC
4. Compute the sum of pairwise Hamming distances of $\{S_1', S_2', \ldots, S_r'\}$
   - If this is at least $d$, stop and return this list
5. For each vertex $v$ in $\bigcup_{i \in [r]} S_i'$
   - Guess the remaining sets to which $v$ belongs
   - Add $v$ to these sets
   - Go to step 4.
6. For each vertex $v \notin \bigcup_{i \in [r]} S_i'$
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   - Go to step 4.
7. Return **NO**

# A simple FPT Algorithm for Diverse Vertex Cover

## Analysis

- Correctness
  - Exhaustive branching

# A simple FPT Algorithm for Diverse Vertex Cover

## Analysis

- Running time
  1. Enumerate all minimal vertex covers MinVC of $G$ of size at most $k$.
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     - $\mathcal{O}^\star(2^{k \cdot r})$ time

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     - $\mathcal{O}^\star(2^{k \cdot r})$ time
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     - Polynomial time

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     - $\mathcal{O}^\star(2^{k \cdot r})$ time
  3. If the sum of pairwise Hamming distances of $\{S_1', S_2', \ldots, S_r'\}$ is at least $d$, stop and return this list
     - Polynomial time
  4. For each vertex $v$ in $\bigcup_{i \in [r]} S_i'$, guess the remaining sets to which $v$ belongs and add $v$ to these sets
     - $\mathcal{O}^\star(2^r)$ time per vertex.
     - Total: $\mathcal{O}^\star(2^{k \cdot r^2})$ time

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     - $\mathcal{O}^\star(2^{k \cdot r})$ time
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     - Polynomial time
  4. For each vertex $v$ in $\bigcup_{i \in [r]} S'_i$, guess the remaining sets to which $v$ belongs and add $v$ to these sets
     - $\mathcal{O}^\star(2^r)$ time per vertex.
     - Total: $\mathcal{O}^\star(2^{k \cdot r^2})$ time
  5. For each vertex $v \notin \bigcup_{i \in [r]} S'_i$, guess the sets to which $v$ belongs and add $v$ to these sets.
     - $\mathcal{O}^\star(2^r)$ time per vertex.
     - The diversity budget $d$ drops by at least $r$ per vertex
     - Depth of branching: $\leq \frac{d}{r}$
     - Total time: $\mathcal{O}^\star(2^{r \cdot \frac{d}{r}}) = \mathcal{O}^\star(2^d) = \mathcal{O}^\star(2^{k \cdot r^2})$

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## Theorem

*Diverse Vertex Cover is FPT and can be solved in $\mathcal{O}^\star(2^{2kr^2+kr})$ time.*

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## Theorem

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## Next steps

1. Can we solve Diverse Vertex Cover faster than this?
   - Yes. Diverse Vertex Cover can be solved in $\mathcal{O}^\star(2^{2kr+2r\log k})$ time.

# A simple FPT Algorithm for Diverse Vertex Cover

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## Next steps

1. Diverse Vertex Cover can be solved in $\mathcal{O}^\star(2^{2kr+2r\log k})$ time.
2. Does Diverse Vertex Cover have a polynomial kernel?

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## Next steps

1. Diverse Vertex Cover can be solved in $\mathcal{O}^\star(2^{2kr+2r\log k})$ time.
2. Does Diverse Vertex Cover have a polynomial kernel?
   - Yes, on $\mathcal{O}(k(k+r))$ vertices

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1. Diverse Vertex Cover can be solved in $\mathcal{O}^\star(2^{2kr+2r\log k})$ time.
2. Diverse Vertex Cover has a kernel on $\mathcal{O}(k(k+r))$ vertices
3. Can we solve diverse variants of other FPT problems in FPT time?
   - E.g: Feedback Vertex Set, (Your favourite problem goes here), ...

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2. Diverse Vertex Cover has a kernel on $\mathcal{O}(k(k+r))$ vertices
3. Can we solve diverse variants of other FPT problems in FPT time?
   - E.g: Feedback Vertex Set, (Your favourite problem goes here), ...
     - Yes, we can!
     - If the problem is FPT parameterized by treewidth

# A Polynomial Kernel for Diverse Vertex Cover

- Input: Graph $G$, integers $k, r, d$
- Parameter: $(k, r)$
- Question: Does $G$ have a list $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ of vertex covers:
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## Reduction Rule

If a vertex $v$ has degree more than $k$ then delete $v$ and decrement $k$.

- Such a vertex $v$ is in *every* solution of size at most $k$
  - It does *not* contribute to the diversity!
  - So: safe to add to every solution

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- If there are isolated vertices in the reduced instance:
  - Keep at most $kr$ of them and delete the rest
  - Let $(G', k')$ be the resulting instance

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- If there are isolated vertices in the reduced instance:
  - Keep at most $kr$ of them and delete the rest
  - Let $(G', k')$ be the resulting instance
- If $G'$ has more than $k(k+1) + kr = k(k+r+1)$ vertices then return a **NO** instance
  - $G'$ has more than $k(k+1)$ vertices of positive degree
- Else: return $(G', k')$

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## Theorem

*Diverse Vertex Cover has a kernel on $k(k+r+1)$ vertices.*

- The same idea works for other problems with a "Buss-style" kernel.
  - Diverse Point Line Cover : on $\mathcal{O}(k(k+r))$ points.
  - Diverse $d$-Hitting Set, for fixed $d$ : on $\mathcal{O}(k^d + kr)$ elements.
  - Diverse Feedback Arc Set in Tournaments : on $\mathcal{O}(k(k+r))$ vertices.
  - …

# An FPT Algorithm for Diverse Feedback Vertex Set

- Input: Graph $G$, integers $k, r, d$
- Parameter: $(k, r)$
- Question: Does $G$ have a list $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ of feedback vertex sets:
  - Each $S_i$ is of size at most $k$, and
  - The sum of pairwise Hamming distances of the $S_i$s is at least $d$

# An FPT Algorithm for Diverse Feedback Vertex Set

## Outline of the Algorithm

1. Compute *one* feedback vertex set $S$ of $G$, of size at most $k$.
   - If there is no such FVS then return **NO**
   - Can be done in $\mathcal{O}^\star(3.619^k)$ time
2. Construct a *nice tree decomposition* of $G$, of width at most $k+1$
   - Can be done in polynomial time
   - For a node $t$ of the decomposition, let:
     - $X_t$ be the bag at $t$
     - $G_t$ be the subgraph "rooted" at $t$
3. For a node $t$, for each tuple $(I_1, I_2, \ldots, I_r \,;\, i_1, i_2, \ldots, i_r)$ where
   - $I_j$ is the intersection of the (unknown) FVS $S_j$ with $X_t$
   - $i_j$ is the size of the intersection of $S_j$ with $V(G_t)$

   store the largest sum of pairwise Hamming distances of any solution $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ which respects the tuple.

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### Theorem

*Diverse Feedback Vertex Set can be solved in* $\mathcal{O}^\star(4.619^k + 2^{2kr+2r\log k})$ *time.*

# Summary

- The diverse version of (essentially) every problem which is
  - FPT when parameterized by treewidth, or
  - FPT by the solution size, and the treewidth can be upper-bounded by the solution size
- The diverse version of (essentially) every problem which has a "Buss-style" polynomial kernel, has a polynomial kernel.

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## Open problems

- Other diversity measures
- Diverse versions of problems not (known to be) FPT parameterized by treewidth
- Hardness results
  - FPT problems whose diverse versions are not FPT?

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Thank You!