Suggestions for master projects

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In the following you will find several suggestions for possible master projects with me. All of them are coming from the context of infinite-dimensional geometry and analysis. Before we start, a few general remarks

1. The projects proposed below require familiarity with infinite-dimensional differential geometry. As this is not (usually) taught at UiB I do not expect you to be familiar with these topics. Instead I hope that you are curious and willing to immerse yourself in the material. Of course, I will provide an introduction and literature at the beginning to help you with the first steps.

2. If none of the projects below appeal to you, but you would like to take a qualification thesis with me anyway lets talk! We can certainly figure something out we will both be interested in.

3. If you have an idea on what you would like to work, lets talk! It should be connected somehow to things[1] I am also interested in though.

In the following you will find several proposals for master projects (in no particular order).

1 Manifolds of mappings with infinite-dimensional target

The set of smooth mappings $C^\infty(M, N)$ between two manifolds M and N appears naturally in many constructions and applications. For example, for $M = \mathbb{S}^1$ the unit circle, the "loop space" $C^\infty(\mathbb{S}^1, N)$ is the space of smooth loops with values in N. These spaces appear naturally in physics applications. Further, $C^\infty(M, N)$ contains many subspaces of interest in differential topology and geometry, e.g. the space of smooth immersions which is used in image and shape analysis, cf. [1]. It is well known that one can endow

[1]Usually that means (infinite-dimensional) geometry and Lie theory, as well as its applications in numerics, stochastic analysis etc., see for example the overview on my webpage https://www.uib.no/en/persons/Alexander.Schmeding

$C^\infty(M,N)$ with a suitable topology[2] and even an (infinite-dimensional) manifold structure such that many natural operations between these manifolds of mappings, such as the pushforward

$$f_*\colon C^\infty(M,N) \to C^\infty(M,L), \quad g \mapsto f \circ g, \quad f \in C^\infty(N,L)$$

become continuous and even smooth. In particular, if $M = N$ the manifold structure of $C^\infty(M,M)$ turns the (smooth) diffeomorphisms of M into a Lie group.

One may wonder now what happens if one iterates the construction and considers a manifold of mappings which takes values in an infinite dimensional manifold, e.g. $C^\infty(M, C^\infty(N,L))$. As long as M stays finite dimensional, the function space topologies still make sense (this was the main result of [2]) and it is well known that the construction of the manifold structure generalises if M is compact.

Aim of the project

Describe the manifold structure of $C^\infty(M,N)$ if M is non-compact and N an infinite-dimensional manifold. Then establish smoothness of the natural function space operations and consider applications of these results in differential geometry.

Expected tasks and work program

1. Learn basic infinite-dimensional calculus and manifolds needed for the project
2. Generalise the classical construction of a manifold of mappings to the new setting (suitable tools for the more general setting exist in the literature!)
3. Establish smoothness of natural operations on function spaces such as pushforward, composition etc. (mostly classical techniques carry over)
4. (Optional) Application (e.g. construct a certain infinite-dimensional Lie group of mappings)

References

[1] Celledoni, E., Eidnes, S. and Schmeding, A, *Shape analysis on homogeneous spaces: a generalised SRVT framework* E. Celledoni et al. (eds.), Computation and Combinatorics in Dynamics, Stochastics and Control, Abelsymposium 2016, Abel Symposia 13, 2019

[2] Hjelle, E. O. and Schmeding, A. *Strong topologies for spaces of smooth maps with infinite-dimensional target*, Expo Math, **35** (2017)(1): 13–53

[3] Michor, P. W, *Manifolds of Differentiable Mappings*, Shiva Mathematics Series, **3** (Shiva Publishing Ltd, Nantwich, 1980)

[2] If M is compact this is the so called compact-open C^∞-topology, which allows one to control a function and up to finitely many of its derivatives on a given compact set.

2 Lagrangian bisections of symplectic Lie groupoids

Symplectic geometry lives on even dimensional spaces and manifolds, and measures the sizes of 2-dimensional objects rather than the 1-dimensional lengths and angles that are familiar from Euclidean and Riemannian geometry. The concept arose in the study of classical (Hamiltonian) mechanical systems, such as a planet orbiting the sun or an oscillating pendulum.[3] In some applications one wants to combine symplectic with group like structures. To this end, Weinstein has introduced symplectic groupoids in [7]: A symplectic groupoid is a manifold with a partially defined multiplication[4] (satisfying certain axioms) and a compatible symplectic structure. Though symplectic groupoids are not groups, one can attach a group of generalised elements, the so-called bisections, to a symplectic groupoid. One can then prove that the bisections form an infinite-dimensional Lie group (for the right groupoid, one obtains for example the diffeomorphism group $\mathrm{Diff}(M)$ as the bisection group).

In general the bisections do not preserve the symplectic structure (in the example, a diffeomorphism φ usually does not preserve the symplectic form, i.e. $\varphi^*\omega = \omega$ does not necessarily hold). Thus one is led to the subgroup of all bisections which preserve the symplectic structure, these are the so-called Lagrangian bisections. Rybicki could show that in a certain setting of infinite-dimensional calculus, the Lagrangian bisections form a Lie group. Unfortunately, the manifold structure Rybicki considers is somewhat non-canonical and in addition we would like these results in a different setting of infinite-dimensional calculus.

Aim of the project

Construct the Lie group of Lagrangian bisections in the so-called Bastiani setting of infinite-dimensional calculus. For this we will consider the Lagrangian bisections as a subgroup of all bisections [6] (with a canonical Lie group structure). Then we adapt Rybicki's proof to our setting.

Expected tasks and work program

1. Learn basic symplectic geometry and familiarise yourself with Lie groupoids and symplectic groupoid.

2. Review the Lie group structures of the bisections and the symplectomorphisms in the Bastiani setting.

3. Understand and adapt the ideas in [4] to construct the Lie group of Lagrangian bisections in the another infinite-dimensional setting

4. (Optional) Describe Lie theoretic properties of the Lagrangian bisections

[3] Compare "What is symplectic geometry?" talk by Dusa McDuff, http://www.math.stonybrook.edu/~dusa/ewmcambrevjn23.pdf.

[4] Think of a set of arrows which you can only compose (=multiply) if the tip of one arrow points at the end of another arrow.

Variant: Legendre bisections of contact groupoids

Similar to the project for Lagrangian bisections of a symplectic groupoid we are interested here in another subgroup of the bisection group. However we change the structure on the groupoid to a contact structure. A contact structure on an odd dimensional manifold is a smoothly varying family of codimension one subspaces of each tangent space of the manifold, satisfying a non-integrability condition.[5] In a certain sense, contact structures form odd dimensional counter parts to the even dimensional symplectic geometry. Again contact groupoids [3] are manifolds with a partial multiplication and a compatible contact structure. Also in this case one can construct a subgroup of (contact) structure preserving bisections, the so-called Legendre bisections. As Rybicki has shown in [5] the Legendre bisections form an infinite-dimensional Lie group (in a certain setting of infinite-dimensional calculus with a non-canonical manifold structure).

Aim of the project

Construct the Lie group of Legendre bisections in the so-called Bastiani setting of infinite-dimensional calculus. For this we will consider the Legendre bisections as a subgroup of all bisections (with a canonical Lie group structure). Then we adapt Rybicki's proof to our setting.

References

[1] Cannas da Silva, A. *Lectures on symplectic geometry* Lecture Notes in Mathematics, 1764. 2001, https://people.math.ethz.ch/~acannas/Papers/lsg.pdf

[2] Geiges, H. *A brief history of contact geometry and topology* Expo. Math. 19, No. 1, 2001 pp. 25–53.

[3] Libermann P. *On symplectic and contact groupoids* Differential geometry and its applications (Opava, 1992), 29–45, Math. Publ., 1, Silesian Univ. Opava, 1993.

[4] Rybicki, T. *On the group of Lagrangian bisections of a symplectic groupoid* in "Lie algebroids and related topics in differential geometry" (Warsaw, 2000), 235–247, Banach Center Publ., 54

[5] Rybicki, T. *On Contact Groupoids and Legendre Bisections* Contemporary Mathematics, Vol. 288, 2001

[6] Schmeding, A. and Wockel C. *The Lie group of bisections of a Lie groupoid* Ann. Global Anal. Geom. 48 (2015), no. 1, 87–123

[7] Weinstein A. *Symplectic groupoids and Poisson manifolds* Bull. Amer. Math. Soc. (N.S.) 16 (1987), no. 1, 101–104

[5] See http://www.map.mpim-bonn.mpg.de/Contact_manifold or [2] for more information.

3 A geometric perspective on stochastic PDEs

Arnold showed in [1] that the motion of the rigid body and the motion of an incompressible, inviscid fluid have the same structure. They can be realised as geodesic equations on infinite-dimensional manifolds. A geodesic equation is an ordinary differential equation. By using Arnold's insight, the local existence and uniqueness of solutions for the Euler equation of an incompressible fluid (which is a non-linear partial differential equation (PDE)) can be established using methods for ordinary differential equations. This was made rigorous in the famous paper [2] by Ebin and Marsden. In the following decades Arnold's trick and the Ebin-Marsden approach have been generalised to a wide class of PDEs, the so-called Euler-Arnold PDEs (see e.g. [3] for a survey). These PDEs arise for example in fluid dynamics, magnetohydrodynamics and as the EPDiff equations from image analysis.

Recently we could show in [4] that the analysis from the deterministic PDE setting generalises to a stochastic version of the Euler equation. Here we consider an additive noise, i.e. we consider an equation

$$\frac{\partial}{\partial t}u = F(t,u) + \dot{W},$$

where W is a suitable Brownian motion and F describes the (deterministic) PDE part. Note that the derivative of Brownian motion does not exist almost everywhere. So the equation needs to be interpreted as a (stochastic) integral equation (the term $\dot{W}$ can be interpreted as "white noise"[6]). The idea of the whole theory is that the deterministic part can be split off from the stochastic part (and thus is solved using the already known techniques). Hence one only has to deal with a stochastic differential equation on an infinite-dimensional manifold, for which a solution theory is available. In [4] we have established the theory only for the Euler equation of an incompressible fluid. However, it is expected that (with minor modifications) the same techniques can be applied to other equations from the Euler-Arnold class of PDEs.

Remark: Although this project deals with PDEs and stochastic equations, it should be noted that one does neither apply classical methods from the analysis of PDEs nor from the analysis of stochastic equations. The idea is to cheat and obtain the solution theory from a suitable reformulation of the equations. (Of course it does not hurt to have knowledge about (stochastic) PDEs)

Aim of the project

Generalise the local existence and uniqueness results for solutions of the stochastic Euler equation to an other equation of the Euler-Arnold class. The first (easiest but still interesting!) example for which this will work is the inviscid Burgers equation. To this end, adapt the toolbox developed in [4] needs to be adapted to the new situation.

[6] In T. Hida: *White Noise Theory and Its Applications* Asia Pacific Mathematics Newsletter Vol. 4, No. 4, 2014, https://pdfs.semanticscholar.org/5e7c/646e348175390f51b1af6302cad0a4410464.pdf one can find a broad discussion of the features and applications of "white noise".

Expected tasks and work program

- Familiarize yourself with the basic theory developed in [4] (includes learning the basics of stochastic analysis)
- Study the deterministic theory of the equation at hand (e.g. inviscid Burgers)[7]
- Adapt the methods from [4] to obtain local existence and uniqueness of solutions to a stochastic version of the deterministic PDE.

References

[1] Arnold V.I. *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications á l'hydrodynamique des fluides parfaits* Ann. Inst. Fourier (Grenoble) 16 (1966), 319–361.

[2] Ebin, D.G. and Marsden, J.E. *Groups of Diffeomorphisms and the Motion of an Incompressible Fluid* Annals of Mathematics Vol. 92, No. 1 (Jul., 1970), pp. 102-163

[3] Khesin B. and Wendt, R. *The geometry of infinite-dimensional groups* Results in Mathematics and Related Areas. 3rd Series. Vol. 51, Springer, 2009

[4] Maurelli, M. and Modin, K. and Schmeding, A. *Incompressible Euler equations with stochastic forcing: a geometric approach*, https://arxiv.org/abs/1909.09982

[7] A nice introduction to the mechanism is B. Kolevs article https://arxiv.org/pdf/math-ph/0402052.pdf.