The Asymptotic Extremal $h$-range for the Postage Stamp Problem with Four Stamp Denominations

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Abstract

In this article we determine the asymptotic extremal $h$-range for the postage stamp problem with four stamp denominations. This asymptotic extremal $h$-range was conjectured by Mossige in 1986. He constructed a basis giving the mentioned $h$-range and in this way he gave a lower bound. Lateron Selmer gave another, in a sense dual, basis with the same $h$-range. The core content of this report is to give an upper bound that actually meets the lower bound thus showing that the Mossige Selmer conjecture was right. The task is to find an upper bound for a real function given a number of linear constraints. This is the easy part. The hard part is to show that all possible constellations of linear systems of constraints have been covered. The report shows the way of organizing the work in detail. One part of the problem had to be solved by a computer. The programme is enclosed.

1 Introduction

A set of $k$ positive integers $A_k = \{a_1 = 1 < a_2 < a_3 \cdots < a_k\} \subseteq \mathbb{N}$ is called a basis. Let now $h \in N, M \in N_0 = \mathbb{N} \cup \{0\}$ be integers. We say $M$ is $h$-representable by $A_k$, if there exist non-negative integer coefficients $x_1, x_2, \ldots, x_k \in \mathbb{N}_0$ such that

$$\sum_{i=1}^{k} x_i a_i = M \text{ and } \sum_{i=1}^{k} x_i \leq h.$$ 

The set of all integers which are $h$-representable by $A_k$ is denoted by $hA_k$. We consider now the least positive integer $N$ without such an $h$-representaion by $A_k$, and call $N - 1$ the $h$-range $n_h(A_k)$ of $A_k$:

$$n_h(A_k) = \min\{n \in \mathbb{N} \mid n \notin hA_k\} - 1.$$ 

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Often we call a basis an \textit{h-range basis}. We may think of the \textit{h-range problem} as a problem concerning stamps on an envelope. Given a set of stamp denominations including 1 we may ask for first fare that cannot be put on an envelope of given size - say enough space for \( h \) stamps. The \textit{h-range} is then the largest number such that all fares from 1 up to this number can be put on the envelope not using more than \( h \) stamps from the set. With \( h_0 \) we denote the least number of addends that is sufficient for the \textit{h-range} to reach the largest element \( a_k \):

\[
h_0 = h_0(A_k) = \min \{ h \in \mathbb{N} \mid n_h(A_k) \geq a_k \}.
\]

It is easy to show that for \( h \geq h_0 - 1 \) we have

\[
n_{h+1}(A_k) \geq a_k + n_h(A_k).
\]

For sufficiently large \( h \) even equality holds:

\[
n_{h+1}(A_k) = a_k + n_h(A_k), \text{ for } h \geq h_1 \geq h_0 - 1. \tag{1}
\]

For a proof see Selmer [17] or Meures [10]. By \( h_1 \) we denote the least bound for (1) to hold.

An \textit{h-representation} of \( M \in \mathbb{N} \) by \( A_k = \{1, a_2, a_3, \ldots, a_k\} \):

\[
\sum_{i=1}^{k} x_i a_i = M, \quad \sum_{i=1}^{k} x_i \leq h,
\]

is called the \textit{regular representation} by \( A_k \), if \( a_k \), the largest element of the basis, is used as often as possible, and then \( a_{k-1} \) is used as often as possible to represent the rest \( M - x_h a_k \), and so on. A representation of \( M \in \mathbb{N} \) is called \textit{minimal} if the number of addends used in the representation is minimal among all possible representations. The regular representation need not always be minimal.

On the other hand, the minimal representation can always be achieved by starting with the regular one and performing several "transfers" of basis elements. In order to describe this process we need the so-called \textit{normal form} of the basis \( A_k \), introduced by Hofmeister [6]. Let \( A_k \) be a basis, then there exist uniquely determined integers \( \gamma_i \geq 2 \) for \( i = 1, 2, \ldots, k - 1 \) and \( \beta_j^{(i)} \) for \( i = 2, 3, \ldots, k \) and \( j = 1, 2, \ldots, i - 2 \) such that

\[
\gamma_{i-1} = \left\lfloor \frac{a_i}{a_{i-1}} \right\rfloor, a_i = \gamma_{i-1} a_{i-1} - \sum_{j=1}^{i-2} \beta_j^{(i)} a_j \text{ for } i = 2, 3, \ldots, k, \tag{2}
\]

where \( \sum_{j=1}^{i-2} \beta_j^{(i)} a_j \) is the regular representation of \( \gamma_{i-1} a_{i-1} - a_i \). Here \( \left\lfloor x \right\rfloor \) denotes the least integer \( \geq x \in \mathbb{R} \).

Let now

\[
M = \sum_{i=1}^{k} e_i a_i
\]
be the regular representation of $M \in \mathbb{N}$, and $s_i \in \mathbb{Z}$ for $i = 2, 3, \ldots, k$. Then

$$M = \sum_{i=1}^{k} e_i a_i + \sum_{i=2}^{k} s_i \left( \gamma_{i-1} a_{i-1} - a_i - \sum_{j=1}^{i-2} \beta^{(i)}_j a_j \right)$$

$$= \sum_{j=1}^{k} \left( e_j - s_j + s_{j+1} \gamma_j - \sum_{i=j+2}^{k} s_i \beta^{(i)}_j \right) a_j,$$

where we put $s_1 = s_{k+1} = \gamma_k = 0$. If we write

$$x_j = e_j - s_j + s_{j+1} \gamma_j + \sum_{i=j+2}^{k} s_i \beta^{(i)}_j \text{ for } j = 1, 2, \ldots, k,$$

we get

$$M = \sum_{i=1}^{k} x_i a_i. \quad (4)$$

Now Hofmeister [6] showed that for every representation - also for the minimal one - there exist non-negative integers $s_i$, $i = 2, 3, \ldots, k$ such that the representation is given by (4) and (3) holds. So every representation of $M \in \mathbb{N}$ is defined uniquely by an integer vector $(s_2, s_3, \ldots, s_k) \in \mathbb{N}_0^{k-1}$. This integer vector is called a transfer or substitution of basis elements.

By the gain $G(s_2, s_3, \ldots, s_k)$ of a transfer $(s_2, s_3, \ldots, s_k) \in \mathbb{N}_0^{k-1}$, we mean the reduction in the number of addends caused by this transfer in comparison to the number of addends in the regular representation:

$$G(s_2, s_3, \ldots, s_k) = \sum_{j=1}^{k} (e_j - x_j) = \sum_{j=1}^{k} \left( s_j - s_{j+1} \gamma_j + \sum_{i=j+2}^{k} s_i \beta^{(i)}_j \right).$$

For the minimal representation of a number $M \in \mathbb{N}$ the gain is always $\geq 0$.

## 2 Extremal Bases

### 2.1 Known Results

Now we fix a number of elements $k$ in our $h$-range basis and ask for the basis $A^*_k = \{1, a^*_1, a^*_2, \ldots, a^*_k\}$ or possibly those bases, with largest $h$-range for a given integer $h$. These bases are called extremal or optimal. Our main interest is not the particular extremal basis but the sequence $A^*_k(h)$ of extremal bases, when $h$, the number of addends allowed in the representations, increases to infinity. Throughout this paper we regard $k$ as a fixed number. Rohrbach [15] could show by a simple combinatorial argument, that for all bases $A^*_k$ there is a common bound for the $h$-range:

$$n_h(A^*_k) < \binom{h+k}{k},$$
and Rødseth [16] was able to sharpen this bound to

\[ n_h(A_k) \leq \frac{(k-1)^{k-1}}{(k-1)!} \left( \frac{h}{k} \right)^k + O(h^{k-1}). \]  

(5)

Since these bounds are of course also valid for the extremal bases \( A^*_k(h) \), we can find a real positive constant \( C \in \mathbb{R} \) such that

\[ n_h(A^*_k(h)) \leq C(h/k)^k. \]

Now it turns out that \((h/k)^k \) already is the right "size" of the extremal \( h \)-range, since Stöhr [20] could show that there exist a real positive constant \( c \in \mathbb{R} \) such that

\[ c(h/k)^k \leq n_h(A^*_k(h)) \leq C(h/k)^k. \]  

(6)

In fact, by Stöhr's result [20] we have \( c \geq 1 \) and by Rødseth's bound (5) we have \( C \leq (k-1)^{k-1}/(k-1)! \).

The author [9] was able to show that the limit

\[ T_k = \lim_{h \to \infty} n_h(A^*_k(h))/(h/k)^k \]

exists for all \( k \). The determination of these limits is called the asymptotic extremal \( h \)-range problem. The hunt for the corresponding coefficient when \( k = 4 \) is the main subject of this article. We will be especially interested in

\[ T_4 = \lim_{h \to \infty} \frac{n_h(A^*_4(h))}{(h/4)^4}. \]

Hofmeister [6] could show that for all parameter bases \( A_k(h) \) that satisfy (6) - and there are of course many more such bases than the extremal ones - the "size" of the basis elements \( a_i(h) \) is given by a simple formula. He showed that there exist real positive constants \( c_i, C_i \in \mathbb{R}, i = 1, 2, \ldots, k \) such that

\[ c_i h^{i-1} \leq a_i(h) \leq C_i h^{i-1}, \text{ for } i = 1, 2, \ldots, k. \]  

(7)

The constants \( c_i \) and \( C_i, i = 1, 2, \ldots, k \) are depending on the number \( k \) of basis elements but not on \( h \). For \( k = 2 \) we know the extremal bases by Stöhr's result [??]. He showed that

\[
\begin{align*}
A^*_2(h) &= \{1, (h+3)/2\}, \text{ if } h \text{ is odd, and} \\
A^*_2(h) &= \{1, (h+3 \pm 1)/2\}, \text{ if } h \text{ is even}
\end{align*}
\]

(8)

(9)

The corresponding extremal \( h \)-range is given by one formula:

\[ n_h(A^*_2(h)) = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor. \]
Here $\lfloor x \rfloor$ denotes the largest integer $\leq x \in \mathbb{R}$.

In 1968 Hofmeister [5] and [7] found out how to determine the extremal $h$-ranges and the corresponding extremal bases in the case $k = 3$. Let

$$\beta(h) = \lfloor \frac{4h + 4}{9} \rfloor, \quad \gamma(h) = \lfloor \frac{2h}{9} \rfloor + 2.$$  

If $h \geq 23$, the extremal bases $A^*_{3}(h)$ are given by

$$a^*_2(h) = 2\beta(h) - \gamma(h) + 1, \quad a^*_3(h) = \gamma(h)a^*_2(h) - \beta(h),$$  

with the corresponding $h$-range

$$n_h(A^*_3(h)) = (h + 4 - \beta(h) - \gamma(h))a^*_3(h) + (\gamma(h) - 2)a^*_2(h) + \beta(h) - 2.$$  

Originally, Hofmeister’s proof was only valid for sufficiently large $h$. A student of his, Hertsch [3], showed that it was enough to claim $h \geq 500$. In a new attempt, Hofmeister could reduce this to $h \geq 200$, and Mossige [11] verified the theorem for $23 \leq h \leq 200$ on a computer. For $h < 23$ the extremal bases and their $h$-ranges can easily be determined by a computer. Table 1 below contains all these bases and the $h$-ranges. Note that for $h = 11$ and $h = 22$ there are two extremal bases.

\begin{center}
\textbf{Table 1.}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\textbf{$h$} & \textbf{a$_2$} & \textbf{a$_3$} & \textbf{$n_h(A^*_3(h))$} & \textbf{$h$} & \textbf{a$_2$} & \textbf{a$_3$} & \textbf{$n_h(A^*_3(h))$} \\
\hline
1 & 2 & 3 & 3 & 12 & 11 & 37 & 212 \\
2 & 3 & 4 & 8 & 13 & 13 & 34 & 259 \\
3 & 4 & 5 & 15 & 14 & 12 & 52 & 302 \\
4 & 5 & 8 & 26 & 12 & 15 & 52 & 534 \\
5 & 6 & 7 & 35 & 16 & 15 & 54 & 418 \\
6 & 7 & 12 & 52 & 17 & 14 & 61 & 476 \\
7 & 8 & 13 & 69 & 18 & 15 & 80 & 548 \\
8 & 9 & 14 & 89 & 19 & 18 & 65 & 633 \\
9 & 9 & 20 & 112 & 20 & 17 & 91 & 714 \\
10 & 10 & 26 & 146 & 21 & 17 & 91 & 805 \\
11 & 10 & 30 & 172 & 22 & 19 & 102 & 902 \\
11 & 10 & 26 & 172 & 22 & 20 & 92 & 902 \\
\hline
\end{tabular}
\end{center}

Quite a lot of people, Braunschädel [2], Hofmeister and Schell [6], Mossige [12] and [13], and Selmer [18] and [19] and the author have spent great effort on the determination of the extremal $h$-ranges and the corresponding extremal bases in the case $k = 4$. Nevertheless the final answer with the $h$-ranges and the extremal bases are still not known, but the asymptotic result, the correct value for the limit $T_4 = \lim_{h \to \infty} n_h(A^*_4(h))/(h/4)^4$ is now known and will be the core content of this article.
According to Hofmeister’s notation \cite{6}, we write the basis elements of $A_4$ in the normal form:

\begin{align*}
a_1 &= 1 \\
a_2 &= \gamma_1 \\
a_3 &= \gamma_2 a_2 - \beta_1^{(3)} \\
a_4 &= \gamma_3 a_3 - \beta_2^{(4)} a_2 - \beta_1^{(4)}
\end{align*}

Hofmeister and Schell presented in \cite{6} a concrete parameter basis with a quite large $h$-range. They put $h = 12d, d \in \mathbb{N}$ and constructed their basis for each value of $d$. By Mrose \cite{14} this fact does not cause any restriction to the general problem, if our interest is the "asymptotic size" of the $h$-range. The normal form of this basis is:

\begin{align*}
a_1 &= 1 \\
a_2 &= 9d - 6 \\
a_3 &= 3da_2 - (5d - 3) \\
a_4 &= 2da_3 - (d - 1)a_2 - (6d - 4).
\end{align*}

They could show that for this choice of $A_4(h)$ we have

\begin{equation}
n_h(A_4(h)) \geq (3d + 6)a_4 = 2(h/4)^4 + O(h^3). \tag{13}
\end{equation}

For a long time the coefficient 2 in front of the $(h/4)^4$ term was believed to be the largest possible for the $h$-range of $A_4(h)$, until Mossige \cite{12} in 1986 by a slight alteration of the Hofmeister-Schell basis \eqref{12} could achieve a coefficient 2.008 instead of 2 in front of the $(h/4)^4$ term in \eqref{13}. The original Mossige basis has very complicated non-rational coefficients. A basis $A_4(h)$ with an $h$-range quite close to the one that is achieved by the Mossige basis, and where the coefficients are rational, is given thus: If we put $h = 2472d, d \in \mathbb{N}$ and

\begin{align*}
a_1 &= 1 \\
a_2 &= 1869d \\
a_3 &= (603d + 3)a_2 - 1031d \\
a_4 &= (392d + 1)a_3 - (193d + 1)a_2 - 1242d.
\end{align*}

Mossige showed that the $h$-range is given by

\begin{equation}
n_h(A_4(h)) \geq 663da_4 + 193da_2 + (1646d - 2) = 2.0080397(h/4)^4 + O(h^3). \tag{15}
\end{equation}

Later on Selmer \cite{19} found another base - in some sense a dual to the Mossige basis - with the same highest coefficient in the $h$-range. For $h = 2472d, d \in \mathbb{N}$
his basis can be written as

\begin{align*}
a_1 &= 1 \\
a_2 &= 1869d \\
a_3 &= (603d + 3)a_2 - 1441d \\
a_4 &= (392d + 1)a_3 - 193da_2 - 826d.
\end{align*}

(16)

If we apply the upper bound (5) to the case \( k = 4 \) we get

\[ n_h(A_4) \leq 4.5(h/4)^4 + O(h^3), \]

so we are left with quite a big gap. In this article we will show that the actual upper bound coincides with the lower bound (15). When Mossige discovered his basis in 1986 he conjectured this basis to be the extremal one. But since also Selmer’s dual basis has the same coefficient for the asymptotic \( h \)-range we call it the Mossige-Selmer-Conjecture. Our task in this article is to show that the upper bound for the extremal \( h \)-range meets the lower bound given by Mossige (in Selmer [19]). In order to establish upper bounds we have developed several techniques that produce such upper bounds effectively, which we are going to present here.

### 3 Preliminary considerations, the case \( k = 3 \)

In this section we want to show some of the techniques we use to arrive at bounds for the coefficient of the largest power of \( h \) in the formula for the \( h \)-range, \( T_k \). In the case \( k = 3 \) this is the coefficient of \( \left( \frac{h}{3} \right)^3 \). Here we assume that we already have found a basis \( A_3(h) \) which gives a coefficient 4/3. The basis (10) given by Hofmeister can be used. We now show that no other basis can have a larger coefficient. We look at a parameter basis \( A_3(h) \). We often leave out the parameter \( h \) of the basis \( A_3(h) \). If \( n_h(A_3(h)) < \frac{4}{3}(h/3)^3 + O(h^2) \), the basis cannot be an extremal one. Therefore we consider only bases \( A_3(h) \) with

\[ n_h(A_3(h)) \geq \frac{4}{3}(h/3)^3 + O(h^2). \]

(18)

We use the normal form (2), and write

\[ n_h(A_3(h)) = \epsilon_3a_3 + \epsilon_2a_2 + \epsilon_1 \]

(19)

for the regular representation of the \( h \)-range of a given parameter basis \( A_3(h) \). Now Hofmeister [6] showed that if

\[ \epsilon_3 + a\gamma_2 + b\gamma_1 \leq h + \delta \]

(20)
for positive constants $a, b \in \mathbb{R}$ and $\delta \in \mathbb{R}$ then
\[
n_h(A_3(h)) < (\epsilon_3 + 1)a_3 \leq \epsilon_3 \gamma_2 \gamma_1 \leq \frac{1}{ab} \left( \frac{h + \delta}{3} \right)^3 + O(h^2) = \frac{1}{ab} \left( \frac{h}{3} \right)^3 + O(h^2),
\]  
(21)
a consequence of the fact that the geometric mean cannot exceed the arithmetic one. This means that if we establish an inequality (20) where $\frac{1}{ab} < \frac{4}{3}$, then the sequence of bases $A_3(h)$ cannot be extremal and can therefore be excluded from further consideration.

Suppose
\[
\epsilon_3 + 5\gamma_2 \leq h + \delta.
\]  
(22)
Now look at the number
\[
M = (\gamma_2 - 2)a_2 + (\gamma_1 - 1) < a_3 < n_h(A_3).
\]  
(23)
Here no $a_3$-transfer is possible, since there are no such elements in the regular representation (23), and this representation must therefore be minimal, giving
\[
\gamma_2 + \gamma_1 \leq h + 3.
\]  
(24)
Now multiplying (24) by $1/2$ and adding it to (22) yields
\[
\epsilon_3 + \frac{11}{2} \gamma_2 + \frac{1}{2} \gamma_1 \leq \frac{3h}{2} + \delta + 3/2
\]
and in analogy with (21) this gives
\[
n_h(A_3(h)) \leq \frac{3^3 \cdot 2^2}{2^3 \cdot 11} \left( \frac{h}{3} \right)^3 + O(h^2) \leq \frac{27}{22} \left( \frac{h}{3} \right)^3 + O(h^2) < \frac{4}{3} \left( \frac{h}{3} \right)^3 + O(h^2).
\]  
(25)
So we have
\[
\epsilon_3 + 5\gamma_2 > h + \delta.
\]  
(26)
This inequality will help us to exclude a vast multitude of cases.
In the sequel we want to present a method that provides inequalities (20). We do this by looking at several "key numbers" and their representation by $A_3$.
These representations will in general contain the variables $h, \epsilon_3$ and $\gamma_2, \gamma_1, \beta^{(3)}_1$ of (2). Combining several representations, we can get rid of $\beta^{(3)}_1$ and thus arrive at an inequality of the form (20).

4 The method of average for $k = 3$

We look first at the number $N_1$ that has the largest regular coefficient sum of all numbers $< n_h(A_3)$ and its regular representation:
\[
N_1 = (\epsilon_3 - 1)a_3 + (\gamma_2 - 2)a_2 + (\gamma_1 - 1).
\]
Now $N_1$ must have an $h$-representation by $A_3$ using the transfer $S^{(1)} = (s_{2}^{(1)}, s_{3}^{(1)})$, where the upper index denotes that the transfer belongs to $N_1$:

$$
N_1 = (\epsilon_3 - s_3^{(1)})a_3 + ((s_3^{(1)} + 1)\gamma_2 - s_2^{(1)} - 2)a_2 + ((s_2^{(1)} + 1)\gamma_1 - s_3^{(1)}\beta_1^{(3)} - 1).
$$

Since the coefficient sum has to be $\leq h$, we have

$$
\epsilon_3 + (s_3^{(1)} + 1)\gamma_2 + (s_2^{(1)} + 1)\gamma_1 - s_3^{(1)}\beta_1^{(3)} \\
\leq h + 6 + s_2^{(1)} + s_3^{(1)} \leq h + \delta,
$$

where $\delta$ is a constant not depending on $h$, since we know by Hofmeister [6] that for bases $A_3$ with (18), $s_{2}^{(1)}$ and $s_{3}^{(1)}$ are bounded independently of $h$. For the reduction $\kappa_1$ of the constant term, we now assume

$$
\kappa_1 = s_3^{(1)}\beta_1^{(3)} - s_2^{(1)}\gamma_1 > 0,
$$

and consider the next "key number", with constant term $\kappa_1 - 1$:

$$
N_2 = (\epsilon_4 - 1)a_3 + (\gamma_2 - 2)a_2 + (\kappa_1 - 1) \\
= (\epsilon_4 - 1)a_3 + (\gamma_2 - 2)a_2 + (s_3^{(1)}\beta_1^{(3)} - s_2^{(1)}\gamma_1 - 1).
$$

Clearly $N_2 < N_1 < n_h(A_3)$ must have an $h$-representation not using $S^{(1)} = (s_{2}^{(1)}, s_{3}^{(1)})$, since otherwise we would get a coefficient $-1$ in the last position. So $N_2$ uses $S^{(2)} = (s_{2}^{(2)}, s_{3}^{(2)}) \neq S^{(1)} = (s_{2}^{(1)}, s_{3}^{(1)})$, and we get

$$
N_2 = (\epsilon_4 - 1 - s_4^{(2)})a_3 + ((s_3^{(2)} + 1)\gamma_2 - s_2^{(2)} - 2)a_2 \\
+ (s_3^{(1)}\beta_1^{(3)} - s_2^{(1)}\gamma_1 + s_2^{(2)}\gamma_1 - s_3^{(2)}\beta_1^{(3)} - 1),
$$

giving

$$
\epsilon_3 + (s_3^{(2)} + 1)\gamma_2 + (s_3^{(1)}\beta_1^{(3)} - s_1^{(1)}\gamma_1) + (s_2^{(2)}\gamma_1 - s_3^{(2)}\beta_1^{(3)}) \\
\leq h + 6 + s_2^{(2)} + s_3^{(2)} \leq h + \delta,
$$

Now we continue to construct $N_i$ in the same way as before. For the reduction in the last position caused by $S^{(i)} = (s_{2}^{(i)}, s_{3}^{(i)})$, we write
\[ \kappa_i = s_3^{(i)} \beta_1^{(3)} - s_2^{(i)} \gamma_1 \]

for \( i = 1, 2, \ldots, l \) and \( \kappa_0 = \gamma_1 \). We stop the process for \( i = l \), when for the first time

\[ \kappa_l = s_3^{(i)} \beta_1^{(3)} - s_2^{(i)} \gamma_1 < 0. \tag{27} \]

Since the reduction \( \kappa_{i+1} \) cannot exceed the constant term \( \kappa_i - 1 \), we have \( \kappa_i > \kappa_{i+1} \) for \( 1 \leq i \leq l - 1 \), so each \( N_i \) needs a new transfer that has not been used earlier. Since there are only finitely many possible transfers - the numbers \( s_2^{(i)} \) and \( s_3^{(i)} \) are bounded independently of \( h \) - in fact we shall show \( s_3 < 4 \) below - and since there is always a transfer satisfying (27), namely \((0, 0)\), the described process has to terminate after a bounded number (independently of \( h \)) of steps. Since the gain of the used transfers has to be nonnegative, it is easy to see that \( S^{(l)} = (0, 0) \). We collect the inequalities for the coefficient sums of \( N_1, N_2, \ldots, N_l \) in an array:

\[
\begin{align*}
\epsilon_3 + (s_3^{(1)} + 1) \gamma_2 + \gamma_1 - \kappa_1 & \leq h + \delta \\
\epsilon_3 + (s_3^{(2)} + 1) \gamma_2 + \kappa_1 - \kappa_2 & \leq h + \delta \\
\vdots & \vdots \\
\epsilon_3 + \gamma_2 + \kappa_{l-1} & \leq h + \delta
\end{align*}
\]

Averaging gives

\[ \epsilon_3 + \left(1 + \frac{\sum_{i=1}^{l} s_3^{(i)}}{l}\right) \gamma_2 + \frac{\gamma_1}{l} \leq h + \delta. \tag{28} \]

The array of inequations is called the main list.

In the sequel we shall characterize the possible transfers that can be used for \( N_1, N_2, \ldots, N_l \), and shall find bounds for \( l \) and \( \sum_{i=1}^{l} s_3^{(i)} \). Thus we get inequalities (20), which we were looking for.

From linje \( i \) in the main list we see that \( \epsilon_3 + (s_3^{(i)} + 1) \gamma_2 + \kappa_{i-1} - \kappa_i \leq h + \delta \).

Neglecting the constant term we get \( \epsilon_3 + (s_3^{(i)} + 1) \gamma_2 \leq h + \delta \). By (26) we therefore know that \( s_3^{(i)} \leq 3 \). Thus there are only few cases to study. In the case \( k = 4 \) the universe of cases is vastly bigger and thus causes much more trouble. Here we have to examine the following eight cases, where \( A \) specifies the set of transfers to be used in the main list.

1. \( A = \{(0, 0)\} \)
2. \( A = \{(0, 0), (0, 1)\} \)
3. \( A = \{(0, 0), (s_2, 2)\} \)
4. \( A = \{(0, 0), (s_2', 3)\} \)
5. \( A = \{(0, 0), (0, 1), (s_2, 2)\} \)
6. \( A = \{(0, 0), (0, 1), (s'_2, 3)\} \)
7. \( A = \{(0, 0), (s_2, 2), (s'_2, 3)\} \)
8. \( A = \{(0, 0), (0, 1), (s_2, 2), (s'_2, 3)\} \)

We now go through all these cases and show that we get upper bounds \( \leq \frac{4}{3} \) for the coefficient of \( \left( \frac{h}{3} \right)^3 \) in every case.

**Case 1.** Here the mainlist is very short and the only line in this list reads

\[ \epsilon_3 + \gamma_2 + \gamma_1 \leq h + \delta. \]  

By (21) the coefficient is \( \leq 1 \), so we are finished with this case.

**Case 2, 3, and 4** Here the mainlist is

\[ \epsilon_3 + (1 + s_3^{(1)}) \gamma_2 + \gamma_1 + \kappa_1 \leq h + \delta \]
\[ \epsilon_3 + \gamma_2 + \kappa_1 \leq h + \delta. \]

Thus the average inequality reads

\[ \epsilon_3 + \left(1 + \frac{s_3^{(1)}}{2}\right) \gamma_2 + \frac{1}{2} \gamma_1 \leq h + \delta. \]  

By (21) the coefficient is \( \leq \frac{4}{2 + s_3^{(1)}} \leq \frac{4}{3} \), so we are finished with these cases, too. Only in case 2 where \( A = \{(0, 0), (0, 1)\} \) we actually get the coefficient \( \frac{4}{3} \).

**Case 5.** Here we get two subcases. First we assume \( \beta_1^{(3)} < \gamma_1/2 \). Then

\[ 0 < \kappa_2 = \beta_1^{(3)} < \kappa_1 = 2\beta_1^{(3)} < \gamma_1. \]

Thus the mainlist reads

\[ \epsilon_3 + 3\gamma_2 + \gamma_1 - 2\beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_3 + 2\gamma_2 + 2\beta_1^{(3)} - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_3 + \gamma_2 + \beta_1^{(3)} \leq h + \delta. \]

The result of the average inequality does not help here. But if we weight the two first lines with weights 1 and 2 and average, we get

\[ \epsilon_3 + \frac{7}{3} \gamma_2 + \frac{1}{3} \gamma_1 \leq h + \delta. \]  

By (21) the coefficient is \( \leq \frac{9}{7} < \frac{4}{3} \), so we are finished with this first subcase.

Now assume \( \beta_1^{(3)} \geq \gamma_1/2 \). Then

\[ 0 \leq \kappa_2 = 2\beta_1^{(3)} - \gamma_1 < \kappa_1 = \beta_1^{(3)} < \gamma_1. \]

Thus the mainlist reads

\[ \epsilon_3 + 2\gamma_2 + \gamma_1 - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_3 + 3\gamma_2 + \beta_1^{(3)} - 2\beta_1^{(3)} + \gamma_1 \leq h + \delta \]
\[ \epsilon_3 + \gamma_2 + 2\beta_1^{(3)} - \gamma_1 \leq h + \delta. \]
The result of the average inequality does not help here either. But if we weight the two last lines with weights 2 and 1 and average, we again get
\[ \epsilon_3 + \frac{7}{3} \gamma_2 + \frac{1}{3} \gamma_1 \leq h + \delta. \tag{32} \]

Again by (21) the coefficient is \( \leq 9/7 < 4/3 \), so we are finished with this last subcase, too.

**Case 6.** Here we get different subcases according to which of the transfers comes first and last in the main list. In any case the average inequality reads
\[ \epsilon_3 + \frac{7}{3} \gamma_2 + \frac{1}{3} \gamma_1 \leq h + \delta. \tag{33} \]

By (21) the coefficient is \( \leq 9/7 < 4/3 \), so we are finished with this case.

**Case 7.** Here again we get different subcases according to which of the transfers comes first and last in the main list. In any case the average inequality reads
\[ \epsilon_3 + \frac{8}{3} \gamma_2 + \frac{1}{3} \gamma_1 \leq h + \delta. \tag{34} \]

By (21) the coefficient is \( \leq 9/8 < 4/3 \), so we are finished with this case.

**Case 8.** Here we get four different subcases according to which of the transfers comes first and last in the main list. Assume first \( \beta_1^{(3)} < \gamma_1/3 \). Then
\[ 0 < \kappa_3 = \beta_1^{(3)} < \kappa_2 = 2\beta_1^{(3)} < \kappa_1 = 3\beta_1^{(3)} < \gamma_1. \]

Thus the mainlist reads
\[
\begin{align*}
\epsilon_3 + 4\gamma_2 + \gamma_1 - 3\beta_1^{(3)} & \leq h + \delta \\
\epsilon_3 + 3\gamma_2 + 3\beta_1^{(3)} - 2\beta_1^{(3)} & \leq h + \delta \\
\epsilon_3 + 2\gamma_2 + 2\beta_1^{(3)} - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_3 + \gamma_2 + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]

The result of the average inequality does not help here either. But if we weight the two first lines with weights 1 and 3 and average, we get
\[ \epsilon_3 + \frac{13}{4} \gamma_2 + \frac{1}{4} \gamma_1 \leq h + \delta. \tag{35} \]

Again by (21) the coefficient is \( \leq 16/13 < 4/3 \), so we are finished with this first subcase.

Next, assume \( \gamma_1/3 \leq \beta_1^{(3)} < \gamma_1/2 \). Then
\[ 0 \leq \kappa_3 = 3\beta_1^{(3)} - \gamma_1 < \kappa_2 = \beta_1^{(3)} \leq \kappa_1 = 2\beta_1^{(3)} < \gamma_1. \]

Thus the mainlist reads
\[
\begin{align*}
\epsilon_3 + 3\gamma_2 + \gamma_1 - 2\beta_1^{(3)} & \leq h + \delta \\
\epsilon_3 + 2\gamma_2 + 2\beta_1^{(3)} - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_3 + 4\gamma_2 + \beta_1^{(3)} - 3\beta_1^{(3)} + \gamma_1 & \leq h + \delta \\
\epsilon_3 + \gamma_2 + 3\beta_1^{(3)} - \gamma_1 & \leq h + \delta.
\end{align*}
\]
The result of the average inequality does not help here either. But if we weight the second and third line with weights 2 and 1 and average, we get
\[ \epsilon_3 + \frac{8}{3} \gamma_2 + \frac{1}{3} \gamma_1 \leq h + \delta. \] (36)
Again by (21) the coefficient is \( \leq \frac{9}{8} < \frac{4}{3} \), so we are finished with this subcase, too.

Next, assume \( \gamma_1/2 \leq \beta_1^{(3)} < 2\gamma_1/3 \). Then
\[ 0 \leq \kappa_3 = 2\beta_1^{(3)} - \gamma_1 < \kappa_2 = \beta_1^{(3)} < \kappa_1 = 3\beta_1^{(3)} - \gamma_1 < \gamma_1. \] Thus the mainlist reads
\[
\begin{align*}
\epsilon_3 + 4\gamma_2 + \gamma_1 - 3\beta_1^{(3)} + \gamma_1 & \leq h + \delta \\
\epsilon_3 + 2\gamma_2 + 3\beta_1^{(3)} - \gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_3 + 3\gamma_2 + \beta_1^{(3)} - 2\beta_1^{(3)} + \gamma_1 & \leq h + \delta \\
\epsilon_3 + \gamma_2 + 2\beta_1^{(3)} - \gamma_1 & \leq h + \delta.
\end{align*}
\]

The result of the average inequality does not help here either. But if we weight the second and third line now with weights 1 and 2 and average, we again get
\[ \epsilon_3 + \frac{8}{3} \gamma_2 + \frac{1}{3} \gamma_1 \leq h + \delta \] (37)
with coefficient \( \leq \frac{9}{8} < \frac{4}{3} \), so we are finished with this subcase, too.

Finally, assume \( 2\gamma_1/3 \leq \beta_1^{(3)} < \gamma_1 \). Then
\[ 0 \leq \kappa_3 = 3\beta_1^{(3)} - 2\gamma_1 < \kappa_2 = 2\beta_1^{(3)} - \gamma_1 < \kappa_1 = \beta_1^{(3)} < \gamma_1. \] Thus the mainlist reads
\[
\begin{align*}
\epsilon_3 + 2\gamma_2 + \gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_3 + 3\gamma_2 + \beta_1^{(3)} - 2\beta_1^{(3)} + \gamma_1 & \leq h + \delta \\
\epsilon_3 + 4\gamma_2 + 2\beta_1^{(3)} - \gamma_1 - 3\beta_1^{(3)} + 2\gamma_1 & \leq h + \delta \\
\epsilon_3 + \gamma_2 + 3\beta_1^{(3)} - 2\gamma_1 & \leq h + \delta.
\end{align*}
\]

The result of the average inequality does not help here either. But if we weight the two last lines with weights 3 and 1 and average, we again get
\[ \epsilon_3 + \frac{13}{4} \gamma_2 + \frac{1}{4} \gamma_1 \leq h + \delta. \] (38)
Again by (21) the coefficient is \( \leq \frac{16}{13} < \frac{4}{3} \), so we are finished with the whole case 8 and we can conclude that \( 4/3 \) is an upper bound for the coefficient \( T_3 \) of \( \left( \frac{h}{3} \right)^3 \) in the formula for the extremal \( h \)-range for three element bases.

Now we have illustrated the methods of average and weighted average and may try to transfer these methods to the case \( k = 4 \).
5 Four stamp denominations, \( k = 4 \)

We look at a parameter basis \( A_4 = A_4(h) \). Also here we often leave out the parameter \( h \) of the basis \( A_4(h) \). If \( n_h(A_4(h)) \leq 2(h/4)^4 + O(h^3) \), the basis cannot be an extremal one. Therefore we consider only bases \( A_4(h) \) with

\[
n_h(A_4(h)) \geq 2(h/4)^4 + O(h^3)
\]

we use the normal form (2), and write

\[
n_h(A_4(h)) = \epsilon_4 a_4 + \epsilon_3 a_3 + \epsilon_2 a_2 + \epsilon_1
\]

for the regular representation of the \( h \)-range of a given parameter basis \( A_4(h) \).

Now Hofmeister [6] showed that if

\[
\epsilon_4 + a_\gamma_3 + b_\gamma_2 + c_\gamma_1 \leq h + \delta
\]

for positive constants \( a, b, c \in \mathbb{R} \) and \( \delta \in \mathbb{R} \) then

\[
n_h(A_4(h)) < (\epsilon_4 + 1) a_4 \leq \epsilon_4 \gamma_3 \gamma_2 \gamma_1 \leq \frac{1}{abc} \left( \frac{h + \delta}{4} \right)^4 + O(h^3) = \frac{1}{abc} \left( \frac{h}{4} \right)^4 + O(h^3),
\]

a consequence of the fact that the geometric mean cannot exceed the arithmetic one. This is called the Hofmeister method. This means that if we establish an inequality (41) where \( \frac{1}{abc} \leq 2 \), then the sequence of bases \( A_4(h) \) cannot be extremal and can therefore be excluded from further consideration. Now given such an equality (41), we may refine the result we can get from (42) by additional information. Look again at (23)

\[
M = (\gamma_2 - 2)a_2 + (\gamma_1 - 1) < a_3 < n_h(A_4).
\]

Here no \( a_4 \) or \( a_3 \) transfers are possible, since there are no such elements in the regular representation (23), and this representation must therefore be minimal, giving

\[
\gamma_2 + \gamma_1 \leq h + 3
\]

Now multiplying (24) by a positive weight \( x \) and adding it to (41) yields

\[
\epsilon_4 + a_\gamma_3 + (b + x) \gamma_2 + (c + x) \gamma_1 \leq (1 + x)h + \delta + 3x
\]

and in analogy with (42) this gives

\[
n_h(A_4(h)) \leq \frac{(1 + x)^4}{a(b + x)(c + x)} \left( \frac{h}{4} \right)^4 + O(h^3).
\]

In order to determine the optimal weight \( x \) we have to find the zeroes of the derivative of the \( h \)-range function. This means that \( x \) has to fit into the following quadratic equation:

\[
4(1 + x)^3(b + x)(c + x) = (1 + x)^4((c + x) + (b + x))
\]
or

\[2x^2 + (3(b + c) - 2)x + 4bc - (b + c) = 0\] (44)

Minimizing the coefficient by the optimal choice of

\[x = \max \left\{ 0, \sqrt{\frac{(3(b + c) - 2)^2}{4} + \frac{b + c - 4bc}{2} - \frac{3(b + c) - 2}{4}} \right\}\]

usually gives a better result than the one we could get from the original inequality (41).

By these means we can show at once that (39) implies

\[
\begin{align*}
\epsilon_4 + 8\gamma_3 &> h + \delta \\
\epsilon_4 + 7\gamma_3 + \frac{1}{6}\gamma_2 &> h + \delta \\
\epsilon_4 + 6\gamma_3 + \frac{1}{3}\gamma_2 &> h + \delta \\
\epsilon_4 + 5\gamma_3 + \frac{1}{2}\gamma_2 &> h + \delta \\
\epsilon_4 + 4\gamma_3 + \frac{4}{5}\gamma_2 &> h + \delta \\
\epsilon_4 + 3\gamma_3 + \frac{6}{5}\gamma_2 &> h + \delta \\
\epsilon_4 + 2\gamma_3 + 2\gamma_2 &> h + \delta \\
\epsilon_4 + \gamma_3 + 5\gamma_2 &> h + \delta
\end{align*}
\]

Here we may put

\[t_7 = 0; t_6 = 1/6; t_5 = 1/3; t_4 = 1/2, t_3 = 4/5, t_2 = 6/5, t_1 = 2, t_0 = 5\]

and write all these inequalities in one form

\[\epsilon_4 + (1 + j)\gamma_3 + t_j\gamma_2 > h + \delta, \quad j = 0, 1, \ldots, 7\] (45)

This is going to be used several times later on.

6 The method of average for \(k = 4\)

We look first at the number \(N_1\) that has the largest regular coefficient sum of all numbers < \(n_k(A_4)\) and its regular representation:

\[N_1 = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (\gamma_1 - 1).\]

Now \(N_1\) must have an \(h\)-representation by \(A_4\) using the transfer

\[S^{(1)} = (s_2^{(1)}, s_3^{(1)}, s_4^{(1)}),\]

where the upper index denotes that the transfer belongs
to $N_1$:

$$N_1 = (\epsilon_4 - 1 - s_4^{(1)})a_4 + ((s_4^{(1)} + 1)\gamma_3 - s_3^{(1)} - 2)a_3$$

$$+((s_3^{(1)} + 1)\gamma_2 - s_4^{(1)}\beta_2^{(4)} - s_2^{(1)} - 2)a_2$$

$$+((s_2^{(1)} + 1)\gamma_1 - s_3^{(1)}\beta_1^{(3)} - s_4^{(1)}\beta_1^{(4)} - 1).$$

Since the coefficient sum has to be $\leq h$, we have

$$\epsilon_4 + (s_4^{(1)} + 1)\gamma_3 + (s_3^{(1)} + 1)\gamma_2 - s_4^{(1)}\beta_2^{(4)} +$$

$$(s_2^{(1)} + 1)\gamma_1 - s_3^{(1)}\beta_1^{(3)} - s_4^{(1)}\beta_1^{(4)}$$

$$\leq h + 6 + s_2^{(1)} + s_3^{(1)} + s_4^{(1)} \leq h + \delta,$$

where $\delta$ is a constant not depending on $h$, since we know by Hofmeister [6] that for bases $A_4$ with $(39), s_2^{(1)}, s_3^{(1)}$ and $s_4^{(1)}$ are bounded independent of $h$.

For the reduction $\kappa_1$ of the constant term, we now assume

$$\kappa_1 = s_3^{(1)}\beta_1^{(3)} + s_4^{(1)}\beta_1^{(4)} - s_2^{(1)}\gamma_1 > 0,$$

and consider the next "key number", with constant term $\kappa_1 - 1$:

$$N_2 = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (\kappa_1 - 1)$$

$$= (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (s_3^{(1)}\beta_1^{(3)} + s_4^{(1)}\beta_1^{(4)} - s_2^{(1)}\gamma_1 - 1).$$

Clearly $N_2 < N_1 < nh(A_4)$ must have an $h$-representation not using $S^{(1)} = (s_2^{(1)}, s_3^{(1)}, s_4^{(1)})$, since otherwise we would get a coefficient $-1$ in the last position. So $N_2$ uses $S^{(2)} = (s_2^{(2)}, s_3^{(2)}, s_4^{(2)}) \neq S^{(1)} = (s_2^{(1)}, s_3^{(1)}, s_4^{(1)})$, and we get

$$N_2 = (\epsilon_4 - 1 - s_4^{(2)})a_4 + ((s_4^{(2)} + 1)\gamma_3 - s_3^{(2)} - 2)a_3$$

$$+((s_3^{(2)} + 1)\gamma_2 - s_4^{(2)}\beta_2^{(4)} - s_2^{(2)} - 2)a_2$$

$$+((s_2^{(2)} + 1)\gamma_1 - s_3^{(2)}\beta_1^{(3)} - s_4^{(2)}\beta_1^{(4)} - 1),$$

giving

$$\epsilon_4 + (s_4^{(2)} + 1)\gamma_3 + (s_3^{(2)} + 1)\gamma_2 - s_4^{(2)}\beta_2^{(4)} +$$

$$(s_3^{(2)}\beta_1^{(3)} + s_4^{(2)}\beta_1^{(4)} - s_1^{(1)}\gamma_1) + (s_2^{(2)}\gamma_1 - s_3^{(2)}\beta_1^{(3)} - s_4^{(2)}\beta_1^{(4)})$$

$$\leq h + 6 + s_2^{(2)} + s_3^{(2)} + s_4^{(2)} \leq h + \delta,$$

Now we continue to construct $N_i$ in the same way as before. For the reduction in the last position caused by $S^{(i)} = (s_2^{(i)}, s_3^{(i)}, s_4^{(i)})$, we write
\[ \kappa_i = s_3^{(i)} \beta_1^{(3)} + s_4^{(i)} \beta_1^{(4)} - s_2^{(i)} \gamma_1 \]

for \( i = 1, 2, \ldots, l \) and \( \kappa_0 = \gamma_1 \). We stop the process for \( i = l \), when for the first time
\[ \kappa_l = s_3^{(l)} \beta_1^{(3)} + s_4^{(l)} \beta_1^{(4)} - s_2^{(l)} \gamma_1 \leq 0. \]  

(46)

Since the reduction \( \kappa_{i+1} \) cannot exceed the constant term \( \kappa_i - 1 \), we have \( \kappa_i > \kappa_{i+1} \) for \( 1 \leq i \leq l - 1 \), so each \( N_i \) needs a new transfer that has not been used earlier. Since there are only finitely many possible transfers - the numbers \( s_2^{(i)}, s_3^{(i)} \) and \( s_4^{(i)} \) are bounded independently of \( h \) - and since there is always a transfer satisfying (46), namely \((0,0,0)\), the described process has to terminate after a bounded number (independently of \( h \)) of steps. We collect the inequalities for the coefficient sums of \( N_1, N_2, \ldots, N_l \) in an array:

\[
\begin{align*}
\epsilon_4 + (s_4^{(1)} + 1) \gamma_3 + (s_3^{(1)} + 1) \gamma_2 - s_1^{(1)} \beta_2^{(4)} + \gamma_1 - \kappa_1 & \leq h + \delta \\
\epsilon_4 + (s_4^{(2)} + 1) \gamma_3 + (s_3^{(2)} + 1) \gamma_2 - s_1^{(2)} \beta_2^{(4)} + \kappa_1 - \kappa_2 & \leq h + \delta \\
\vdots & \vdots \\
\epsilon_4 + (s_4^{(l)} + 1) \gamma_3 + (s_3^{(l)} + 1) \gamma_2 - s_1^{(l)} \beta_2^{(4)} + \kappa_{l-1} - \kappa_l & \leq h + \delta 
\end{align*}
\]

Averaging gives
\[
\epsilon_4 + \left(1 + \frac{\sum_{i=1}^{l} s_4^{(i)}}{l}\right) \gamma_3 + \left(1 + \frac{\sum_{i=1}^{l} s_3^{(i)}}{l}\right) \gamma_2 - \frac{\sum_{i=1}^{l} s_4^{(i)}}{l} \beta_2^{(4)} + \frac{\gamma_1}{l} \leq h + \delta, \tag{47}
\]

where we have used that \( \gamma_1 - \kappa_l \geq \gamma_1 \) by (46). The array of inequations is called the main list.

We have seen that the reductions \( \kappa_i \) of the constant term form a decreasing sequence. Also the gains \( G(S^{(i)}) \) of the corresponding transfers form a decreasing sequence. If \( j > i \) we have \( \kappa_j < \kappa_i \) so the transfer \( S^{(i)} \) gives non-negative coefficients in the representation of \( N_i \) and thus is legal. But the transfer \( S^{(i)} \) is preferred in line \( i \), so the gain \( G(S^{(i)}) \) has to exceed the gain \( G(S^{(j)}) \) of the latter transfer, and we have:

\[
G(S^{(1)}) > G(S^{(2)}) > G(S^{(3)}) > \cdots > G(S^{(l)}) \geq 0
\]

In the sequel we shall characterize the possible transfers that can be used for \( N_1, N_2, \ldots, N_l \), and shall find bounds for \( l, \sum_{i=1}^{l} s_4^{(i)}, \sum_{i=1}^{l} s_3^{(i)} \) and \( \beta_2^{(4)} \). Thus we get inequalities (41), which we were looking for.

## 7 The set of possible transfers

The number \( N_i \) in our list has got the regular representation
\[
N_i = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + (\kappa_{i-1} - 1),
\]
since \(0 < \kappa_{i-1} < \gamma_1\) for \(i = 1, 2, \ldots, l\). For this kind of numbers, only few transfers are possible if we claim (39). Assume \(N_i\) uses \(S = (s_2, s_3, s_4)\) in order to achieve the minimal representation - here we leave out the upper index for a moment. Then as usual

\[
N_i = (\epsilon_4 - 1 - s_4)a_4 + ((s_4 + 1)\gamma_3 - s_3 - 2)a_3 + ((s_3 + 1)\gamma_2 - s_4\beta_2^{(4)} - s_2 - 2)a_2 + \kappa_{i-1} - \kappa_i - 1. \tag{48}
\]

and

\[
\epsilon_4 + (s_4 + 1)\gamma_3 + (s_3 + 1)\gamma_2 - s_4\beta_2^{(4)} + \kappa_{i-1} - \kappa_i \leq h + \delta. \tag{49}
\]

Here we find at once that

\[
0 \leq s_4 \leq 6,
\]

since \(s_4 \geq 7\) together with (50) contradicts (45) for \(j = 7\). For \(0 \leq s_4 \leq 6\), (50) together with (45) for \(j = s_4\) implies

\[
0 < (1 + s_3)\gamma_2 - s_4\beta_2^{(4)} < t_{s_4}\gamma_2, \tag{51}
\]

where the left inequality stems from the non-negativity of the coefficients in (48). Now for \(3 \leq s_4 \leq 6\) we have \(t_{s_4} < 1\), and therefore (51) implies

\[
\frac{s_4\beta_2^{(4)}}{\gamma_2} - 1 < s_3 < \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor \quad \text{or} \quad s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor,
\]

if there is a solution of (51) at all. Once \(3 \leq s_4 \leq 6\) is chosen, \(s_3\) is already determined uniquely, if (51) is soluble. Otherwise \(S = (s_2, s_3, s_4)\) cannot be used in (48). For \(s_4 = 1, 2\) we have \(t_1 = 2, t_2 = 1.2,\) so (51) gives

\[
s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor \quad \text{or} \quad s_3 = \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor + 1.
\]

The second alternative arises only when (51) has two solutions.

In the remaining case \(s_4 = 0\), we have by (50)

\[
\epsilon_4 + \gamma_3 + (1 + s_3)\gamma_2 \leq h + \delta.
\]

By (45) for \(j = 0\) it follows that \(0 \leq s_3 \leq 3\) if \(s_4 = 0\).

What about the choice of \(s_2\)? For the minimal representation (48), we always have

\[
0 \leq \kappa_{i-1} - \kappa_i - 1 = s_2\gamma_1 - s_3\beta_1^{(3)} - s_4\beta_1^{(4)} + \kappa_{i-1} - 1 < \kappa_{i-1} - 1,
\]

if \(i < l\). Since \(0 < \kappa_{i-1} \leq \gamma_1\) this means that

\[
\frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1} - 1 < \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)} - \kappa_{i-1} + 1}{\gamma_1} \leq s_2 < \frac{s_3\beta_1^{(3)} + s_4\beta_1^{(4)}}{\gamma_1}.
\]
Thus $s_2$ is uniquely determined

$$s_2 = \left\lfloor \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rfloor$$

for $i < l$.

In the following chapters we use the following abbreviation

$$s_2 = \left\lceil \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rceil = [s_3, s_4]$$

for $i < l$.

F.ex we write $[1, 2]$ for $[\frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1}]$. Now if $i = l$ then $\kappa_l \leq 0$ and we have

$$\kappa_{l-1} - 1 \leq \kappa_{l-1} - \kappa_l - 1 = s_2 \gamma_1 - s_3 \beta_3^{(3)} - s_4 \beta_4^{(4)} + s_3 \beta_3^{(3)} + \kappa_{l-1} - 1 \leq \gamma_1 - 1.$$

giving

$$\frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \leq s_2 \leq \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)} - s_3 \beta_3^{(3)} + \kappa_{l-1} + \gamma_1}{\gamma_1} < \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} + 1$$

thus

$$s_2 = \left\lceil \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rceil$$

for the transfer in the final line of the array. This formula also includes the "transfer" $(0, 0, 0)$. We will use the abbreviation

$$s_2 = \left\lceil \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rceil = [s_3, s_4].$$

We collect all the possible transfers in four sets:

$$A = \left\{(s_2, s_3, s_4) \mid 1 \leq s_4 \leq 6, s_3 = \left\lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \right\rfloor, s_2 = \left\lceil \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rceil \right\}$$

$$B = \left\{(s_2, s_3, s_4) \mid 1 \leq s_4 \leq 2, s_3 = \left\lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \right\rfloor + 1, s_2 = \left\lceil \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rceil \right\}$$

$$C = \left\{(s_2, s_3, s_4) \mid 0 \leq s_4 \leq 6, s_3 = \left\lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \right\rfloor, s_2 = \left\lceil \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rceil \right\}$$

including $(0, 0, 0)$

$$D = \left\{(s_2, s_3, 0) \mid 1 \leq s_3 \leq 3, s_2 = \left\lceil \frac{s_3 \beta_3^{(3)}}{\gamma_1} \right\rceil \right\}$$

The possibilities

$$s_3 = \left\lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \right\rfloor + 1, s_2 = \left\lceil \frac{s_3 \beta_3^{(3)} + s_4 \beta_4^{(4)}}{\gamma_1} \right\rceil$$
cannot be combined if \((s_2, s_3, s_4) \neq (0, 0, 0)\). In this case the gain of the transfer would be negative. Since then \(s_2 \gamma_1 \geq s_3 \beta_1^{(3)} + s_4 \beta_1^{(4)}\), we have

\[
G(s_2, s_3, s_4) = s_4 \beta_1^{(4)} + \beta_2^{(4)} - \gamma_3 + 1 + s_3 (\beta_1^{(3)} - \gamma_2 + 1) + s_2 (1 - \gamma_1)
\]

\[
= (s_4 \beta_1^{(4)} + s_3 \beta_1^{(3)} - s_2 \gamma_1) + (s_4 \beta_2^{(4)} - s_3 \gamma_2) + (s_4 + s_3 + s_2 - s_4 \gamma_3)
\]

\[
< s_4 + s_3 + s_2 - s_4 \gamma_3 < 0
\]

for large \(h\), because the \(s_i\) are bounded, and by (7) \(\gamma_3\) increases when \(h\) increases. By the same argument, we cannot have \(s_2 = \left\lceil \frac{s_3 \beta_1^{(3)} + s_4 \beta_1^{(4)}}{\gamma_3} \right\rceil\) for transfers where \(s_4 = 0\) except for \((0, 0, 0)\). A transfer from \(C\) is assumed to stand at the end of the list for \(N_i\), since only for those transfers \(s_3 \beta_1^{(3)} + s_4 \beta_1^{(4)} - s_2 \gamma_1 \leq 0\) whilst transfers from \(A, B, D\) have to be used earlier in the list.

Now we introduce 11 variables \(r_1, r_2, r_3, r_4, r_5, r_6, q_1, q_2, d_1, d_2, d_3\) taking values from \(\{0, 1\}\), indicating whether the corresponding transfer is used for some \(N_i\) or not. Here \(r_j\) stands for the transfer \((s_2, s_3, j) \in A\), \(q_j\) for \((s_2, s_3, j) \in B\) and \(d_j\) for \((s_2, j, 0) \in D\). In addition we introduce \(s \in \{0, 1, \ldots, 6\}\) for \((s_2, s_3, s) \in C\) used at the end of the list. We then choose values for \(r_j, q_j, d_j\) and \(s\). All possible lists of representations for the \(N_i\) using these corresponding transfers under consideration will give rise to the same average inequality (47), so the ordering of the used transfers within the list does not play any role for us. In fact this is the crucial advantage of our “average method”. Regarding all different orderings would make the problem very large and more difficult to manage.

Bounds for the values of the interesting magnitudes can now be computed:

\[
l = \sum_{j=1}^{6} r_j + \sum_{j=1}^{2} q_j + \sum_{j=1}^{3} d_j + 1,
\]

the last 1 standing for the ultimate line in the list. Further

\[
\sum_{j=1}^{l} s^{(i)}_4 = \sum_{j=1}^{6} jr_j + \sum_{j=1}^{2} jq_j + s,
\]

and

\[
\sum_{j=1}^{l} s^{(j)}_3 = \sum_{j=1}^{6} \left\lfloor \frac{j \beta_2^{(4)}}{\gamma_2} \right\rfloor r_j + \sum_{j=1}^{2} \left( \left\lfloor \frac{j \beta_2^{(4)}}{\gamma_2} \right\rfloor + 1 \right) q_j + \sum_{j=1}^{3} j d_j + \left\lfloor \frac{s \beta_2^{(4)}}{\gamma_2} \right\rfloor.
\]

Since by (2) \(0 \leq \beta_2^{(4)}/\gamma_2 < 1\), we divide the interval \([0, 1]\) into the following 12 smaller intervals and choose \(\beta_2^{(4)}/\gamma_2\) from one of them:

\[
I_1 = [0, \frac{1}{6}) \quad I_2 = [\frac{1}{6}, \frac{1}{4}) \quad I_3 = [\frac{1}{4}, \frac{1}{3}) \quad I_4 = [\frac{1}{3}, \frac{1}{2}) \quad I_5 = [\frac{1}{2}, \frac{2}{3}) \quad I_6 = [\frac{2}{3}, \frac{1}{2})
\]

\[
I_7 = [\frac{1}{2}, \frac{2}{3}) \quad I_8 = [\frac{2}{3}, \frac{3}{4}) \quad I_9 = [\frac{3}{4}, \frac{4}{5}) \quad I_{10} = [\frac{4}{5}, \frac{1}{2}) \quad I_{11} = [\frac{1}{2}, \frac{2}{5}) \quad I_{12} = [\frac{2}{5}, 1).
\]
Now the intervals $I_p, p = 1, 2, \ldots, 12$ are chosen in such a way that all of the values $\lfloor j \beta_2^{(4)}/\gamma_2 \rfloor$ for $j = 1, 2, \ldots, 6$ are constant over the whole of each $I_p$, and so $\sum_{i=1}^l s_3^{(i)}$ is constant over $I_p$. If we put $I_p = [w_p, z_p)$ and $\beta_2^{(4)} \in I_p$, we have $\lfloor j \beta_2^{(4)}/\gamma_2 \rfloor = \lfloor j w_p \rfloor$. Then

$$\sum_{i=1}^l s_3^{(i)} = \sum_{j=1}^6 \lfloor j w_p \rfloor + \sum_{j=1}^2 (\lfloor j w_p \rfloor + 1) q_j + \sum_{j=1}^3 j d_j + \lfloor s w_p \rfloor$$

and

$$\sum_{i=1}^l s_4^{(i)} \beta_2^{(4)} \leq \sum_{i=1}^l s_4^{(i)} z_p \gamma_2,$$

and we have computed bounds for all the values we are interested in, $\sum_{i=1}^l s_3^{(i)}, \sum_{i=1}^l s_4^{(i)}$ and $\sum_{i=1}^l s_4^{(i)} \beta_2^{(4)}$. For each choice of the variables $r_j, q_j, d_j$ and $s$, (47) now gives us an inequality (41). We combine this inequality with (24) as we did before, and get by (43) a bound for the $h$-range coefficient. Altogether we have to consider "only" $12 \cdot 7 \cdot 2^{11}$ cases, since $\beta_2^{(4)}$ is chosen from one of the 12 intervals $I_p, p = 1, 2, \ldots, 12$.

Now we try to reduce the number of cases to consider. Remember that we have chosen $\beta_2^{(4)} \in I_p = [w_p, z_p)$, and can find out whether (51) is soluble or not. If

$$1 + \lfloor s_4 w_p \rfloor - s_4 z_p \geq t_{s_4},$$

then

$$(1 + s_3) \gamma_2 - s_4 \beta_2^{(4)} \geq t_{s_4} \gamma_2$$

throughout $I_p$, and the corresponding transfer $S = (s_2, s_3, s_4) \in A$ or $C$ cannot be used, a fact that reduces the amount of work considerably. In the same way $S = (s_2, s_3, s_4) \in B$ can be excluded if

$$2 + \lfloor s_4 w_p \rfloor - s_4 z_p \geq t_{s_4},$$

We use the latter inequalities to exclude the use of $r_6 = 1$ for $N_i$ from a number of intervals. First we look at $\beta_2^{(4)}/\gamma_2 \in I_2 \cup I_3$ i.e. $1/6 \leq \beta_2^{(4)}/\gamma_2 < 1/4$. Here $s_4 = 6$ implies $s_3 = 1$ and therefore

$$(1 + s_3) \gamma_2 - s_4 \beta_2^{(4)} = 2 \gamma_2 - 6 \beta_2^{(4)} \geq (2 - 6/4) \gamma_2 = \gamma_2/2 \geq t_6 \gamma_2$$

contradicting (45). Thus $r_6 = 0$ when we look for transfers for $N_i$ when $\beta_2^{(4)}/\gamma_2 \in I_2 \cup I_3$. For $I_5$ we have $1/3 \leq \beta_2^{(4)}/\gamma_2 < 2/5$, thus $s_4 = 6$ implies $s_3 = 2$ and therefore we get

$$(1 + s_3) \gamma_2 - s_4 \beta_2^{(4)} = 3 \gamma_2 - 6 \beta_2^{(4)} \geq (3 - 12/5) \gamma_2 = 3 \gamma_2/5 \geq t_6 \gamma_2$$
also this time contradicting (45). For $I_7$ we have $1/2 \leq \beta_2^{(4)}/\gamma_2 < 3/5$, thus $s_4 = 6$ implies $s_3 = 3$ and therefore we get

$$(1 + s_3)\gamma_2 - s_4\beta_2^{(4)} = 4\gamma_2 - 6\beta_2^{(4)} \geq (4 - 18/5)\gamma_2 = 2\gamma_2/5 \geq t_6\gamma_2$$

again in contradiction with (45). For $I_9 \cup I_{10}$ we have $2/3 \leq \beta_2^{(4)}/\gamma_2 < 4/5$, thus $s_4 = 6$ implies $s_3 = 4$ and therefore we get

$$(1 + s_3)\gamma_2 - s_4\beta_2^{(4)} = 5\gamma_2 - 6\beta_2^{(4)} \geq (5 - 24/5)\gamma_2 = 2\gamma_2/5 \geq t_6\gamma_2$$

also this time contradicting (45). Thus we have shown that $r_6 = 1$ cannot occur for any $N_i$ in the intervals $I_2, I_3, I_5, I_7, I_9, I_{10}$. Later on we will show that $r_6 = 0$ also in the remaining intervals.

Here is another way of excluding cases from consideration. Assume we have chosen a number of transfers to form the list of the $N_i$-representations. We consider $N_i$ and $N_j$ and by (50) we get the truncated inequalities

$$
\epsilon_4 + (s_4^{(i)} + 1)\gamma_3 + (s_3^{(i)} + 1)\gamma_2 - s_4^{(i)}\beta_2^{(4)} \leq h + \delta
$$
$$
\epsilon_4 + (s_4^{(j)} + 1)\gamma_3 + (s_3^{(j)} + 1)\gamma_2 - s_4^{(j)}\beta_2^{(4)} \leq h + \delta
$$

By choosing positive weights $x \in \mathbb{R}$ and $(1 - x)$ we can combine the inequalities in the following way:

$$
\epsilon_4 + (xs_4^{(i)} + (1-x)s_4^{(j)} + 1)\gamma_3 + (xs_3^{(i)} + (1-x)s_3^{(j)} + 1)\gamma_2 - (xs_4^{(i)} + (1-x)s_4^{(j)})\beta_2^{(4)} \leq h + \delta.
$$

This combination together with (43) often helps to exclude the case. The situation where $s > 0$ means that there is a transfer with positive gain, which makes the last coefficient in a representation increase, whilst the second coefficient decrease. This looks very odd and usually "good" bases - giving large coefficients for the $h$-ranges - do not behave like this. For good bases the regular representation ($s = 0$) stands at the end of the list. We therefore study the case $s = 0$ first and look at $s > 0$ at the end of this article.

We conclude this section with an overview over the results given by a computer run of the described program for $s = 0$, showing the maximal coefficient bound occurring in the intervals $I_p$, yielding the following situation:

Table 2.
Here we have used a result to be shown in chapter 10, where we manage to show that \( r_6 = 0 \) in all the intervals \( I_p \) for \( p = 2, 3, \ldots, 12 \) and that \( r_5 = r_4 = 0 \) in interval \( I_{12} \). In interval \( I_1 \) no cases with a coefficient above 2.008 were found.

### 8 The method of weighted average

As table 2 shows the method of average leaves us with a number of cases where we did not manage to find an upper bound for the coefficient in front of \( (h/4)^4 \) in the expression for the \( h \)-range below 2.008, meaning that these cases still are candidates for extremal \( h \)-range bases. We now refine this method in order to get better results and in order to exclude many cases from consideration. An example from interval 2 will show us how to proceed. In table 3 we can find the eight cases in \( I_2 \), where the coefficient bound for the \( h \)-range is not below 2.008.

<table>
<thead>
<tr>
<th>Case</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( r_3 )</th>
<th>( r_4 )</th>
<th>( r_5 )</th>
<th>( r_6 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
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<th>( b )</th>
<th>( c )</th>
<th>Coefficient bound</th>
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<td>2.151992</td>
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Table 3. Interval 2: Sets of transfers with coefficient above 2.008
The first of the eight cases in interval 2 contains four substitutions: \( r_1 = 1, r_2 = 1, r_3 = 1 \) and \( s = 0 \) meaning that the following transfers were used: \( (0, 0, 1), (s_2, 0, 2), (s'_2, 0, 3), (0, 0, 0) \). The main list therefore has four lines concluding with a regular representation. Now we know \( 0 \leq \beta_1^{(4)} / \gamma_1 < 1 \). We now divide the interval \([0, 1]\) for the magnitude \( \beta_1^{(4)} / \gamma_1 \) into the smaller intervals \( J_1 = [0, \frac{1}{3}], J_2 = [\frac{1}{3}, \frac{1}{2}], J_3 = [\frac{1}{2}, \frac{2}{3}] \) and \( J_4 = [\frac{2}{3}, 1] \). If we choose \( \beta_1^{(4)} / \gamma_1 \in J_4 \), we have

\[
k_1 = \beta_1^{(4)} > k_2 = 2\beta_1^{(4)} - \gamma_1 > k_3 = 3\beta_1^{(4)} - 2\gamma_1 > k_4 = 0
\]

and we can write down the corresponding inequality system

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + 3\beta_1^{(4)} - 2\gamma_1 &\leq h + \delta
\end{align*}
\]

Here \( k_1 - k_2 = k_2 - k_3 = \gamma_1 - \beta_1^{(4)} \) whilst \( k_3 - k_4 = k_3 = 3\beta_1^{(4)} - 2\gamma_1 \). We now weight the lines of the inequality system in such a way that \( \beta_1^{(4)} \) vanishes (\( \beta_1^{(3)} \) did not occur in the inequality system). Of course the weights have to be non-negative. Many weight distributions may be possible, but the distribution 0, 0, 3 and 1 gives a good result.

\[
\epsilon_4 + \frac{13\gamma_3}{4} + \gamma_2 - \frac{9\beta_2^{(4)}}{4} + \frac{\gamma_1}{4} \leq h + \delta.
\]

Using the bound for \( \beta_2^{(4)} < \gamma_2 / 5 \) we get

\[
\epsilon_4 + \frac{13\gamma_3}{4} + \frac{11\gamma_2}{20} + \frac{\gamma_1}{4} \leq h + \delta
\]

a linear inequality with the quantities \( \epsilon_4, \gamma_3, \gamma_2 \) and \( \gamma_1 \) on the left side and by (41) and (43) we are able to give an upper bound for the coefficient in front of \( (h/4)^4 \) for the \( h \)-range. In this case the coefficient bound is \( 1.87 < 2.008 \) and the case may be excluded from further consideration.

Now we examine the case where \( \beta_1^{(4)} / \gamma_1 \in J_3 \). This means

\[
k_1 = 3\beta_1^{(4)} - \gamma_1 > k_2 = \beta_1^{(4)} > k_3 = 2\beta_1^{(4)} - \gamma_1 > k_4 = 0
\]

and again we can write down the corresponding inequality system

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 2\gamma_1 - 3\beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - \gamma_1 &\leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + 2\beta_1^{(4)} - \gamma_1 &\leq h + \delta
\end{align*}
\]
Using the weight distribution 0, 1, 2 and 0 gives
\[ \epsilon_4 + \frac{8\gamma_3}{3} + \gamma_2 - \frac{5\beta_2^{(4)}}{3} + \frac{\gamma_1}{3} \leq h + \delta. \]
Using the upper bound for \( \beta_2^{(4)} < \gamma_2/5 \) we get
\[ \epsilon_4 + \frac{8\gamma_3}{3} + \frac{2\gamma_2}{3} + \frac{\gamma_1}{3} \leq h + \delta, \]
yielding the upper bound 1.65 for the coefficient and the case is excluded.
Next we examine the case where \( \beta_1^{(4)}/\gamma_1 \in \mathcal{J}_2 \) giving the following ordering of \( \kappa_i \):
\[ \kappa_1 = 3\beta_1^{(4)} > \kappa_2 = \beta_1^{(4)} > \kappa_3 = 3\beta_1^{(4)} - \gamma_1 > \kappa_4 = 0 \]
and again we can write down the corresponding inequality system
\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + 3\beta_1^{(4)} - \gamma_1 & \leq h + \delta
\end{align*}
\]
Using the weight distribution 0, 2, 1 and 0 gives the same inequality and coefficient bound as in the previous case.
Finally we examine the case where \( \beta_1^{(4)}/\gamma_1 \in \mathcal{J}_1 \) giving the following ordering of \( \kappa_i \):
\[ \kappa_1 = 3\beta_1^{(4)} > \kappa_2 = 2\beta_1^{(4)} > \kappa_3 = \beta_1^{(4)} > \kappa_4 = 0 \]
and again we can write down the corresponding inequality system
\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + \beta_1^{(4)} & \leq h + \delta
\end{align*}
\]
Using the weight distribution 1, 3, 0 and 0 gives the same inequality and coefficient bound as in the first case. This concludes the consideration of the first case in \( \mathcal{I}_2 \) and we see that the effect of the weighted average method exceeds the effect of the unweighted average method. Unfortunately using the weighted average method claims that we know the ordering of the transfers, an information that was not necessary for the unweighted average method. One case for the unweighted average method splits here into four cases for the weighted average method, so we must be prepared that the amount of work increases when we go from the unweighted to the weighted average method.
In the same manner as shown above we can proceed and find upper bounds below 2.008 for all the other cases in $I_2$ except the last one, where $r_1 = 1, r_2 = 1, r_3 = 1, r_4 = 1, r_5 = 1$ and $s = 0$. Here the interval $[0, 1)$ for $\beta_1^{(4)}/\gamma_1$ had to be divided in the following subintervals:

$J_1 = [0, \frac{1}{5}), J_2 = [\frac{1}{5}, \frac{1}{4}), J_3 = [\frac{1}{4}, \frac{1}{3}), J_4 = [\frac{1}{3}, \frac{2}{5}), J_5 = [\frac{2}{5}, \frac{1}{2}), J_6 = [\frac{1}{2}, \frac{3}{5}), J_7 = [\frac{3}{5}, \frac{2}{3}), J_8 = [\frac{2}{3}, \frac{3}{4}), J_9 = [\frac{3}{4}, \frac{4}{5})$ and $J_{10} = [\frac{4}{5}, 1)$. Thus 10 orderings have to be studied. For two of them the computer did not manage to find an upper bound below 2.008.

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 5\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + 5\beta_1^{(4)} - 4\gamma_1 &\leq h + \delta
\end{align*}
\]

and

\[
\begin{align*}
\epsilon_4 + 6\gamma_3 + \gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + \beta_1^{(4)} &\leq h + \delta
\end{align*}
\]

In both situations we get the coefficient 2.1022, not enough to exclude them. Here new techniques are needed to bring us further. These are described in the next section.

We conclude this section with an overview over the results the method of weighted average can contribute for the particular intervals:

Table 4. Number of inequality systems with coefficient above 2.008
Interval & Inequality systems with coefficient bound \\
\text{Interval} I_p & \text{Inequality systems with coefficient bound} \\
I_1 = [0, \frac{1}{9}) & > 2.008 \\
I_2 = [\frac{1}{9}, \frac{1}{9}) & 0 \\
I_3 = [\frac{1}{9}, \frac{1}{9}) & 2 \\
I_4 = [\frac{1}{9}, \frac{1}{9}) & 36 \\
I_5 = [\frac{1}{9}, \frac{1}{9}) & 37 \\
I_6 = [\frac{1}{9}, \frac{1}{9}) & 44 \\
I_7 = [\frac{1}{9}, \frac{1}{9}) & 354 \\
I_8 = [\frac{1}{9}, \frac{1}{9}) & 53 \\
I_9 = [\frac{1}{9}, \frac{1}{9}) & 93 \\
I_{10} = [\frac{1}{9}, \frac{1}{9}) & 116 \\
I_{11} = [\frac{1}{9}, \frac{1}{9}) & 84 \\
I_{12} = [\frac{1}{9}, 1) & 387 \\

9 Key numbers of higher order

Since the information in (53) and (54) is not sufficient to exclude these cases we have to look for additional information. In this case it is sufficient to study a new key number

\[ M(5) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (5\beta_2^{(4)} - 1)a_2 + (\gamma_1 - 1) \]

and the corresponding interval \( K(5) = [M(5) - \gamma_1 + 1, M(5)] \). The key number \( M(5) \) is chosen in such a way that the transfer with the largest reduction \( 5\beta_2^{(4)} \) of the \( a_2 \)-coefficient in (53) or (54) cannot be used. Otherwise we would get a negative coefficient at the second place (in front of \( a_2 \)). Now we hope to get new information from the representations of \( n \in K(5) \). Therefore we have to find out which transfers may be used in \( K(5) \). In general we look at

\[ M(j) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (j\beta_2^{(4)} - \left\lfloor \frac{j\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 - 1)a_2 + (\gamma_1 - 1) \]

and \( K(j) = [M(j) - \gamma_1 + 1, M(j)] \). Of course the transfers from \( A, B, C \) and \( D \) are candidates with the exception of \((s_2, s_3, j) \in A, \) but there may be more. Of course \((s_2, s_3, s_4) \) with \( s_4 \geq 7 \) is impossible by the same reason as for \( N_1 \) and therefore \( A \) and \( C \) cannot be extended. But in the case of \( B \) and \( D \) we have to have a second look.

For \( 0 \leq s_4 \leq 6 \) we have

\[ 0 < j\beta_2^{(4)} - \left\lfloor \frac{j\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 - s_4\beta_2^{(4)} + \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 < t_{s_4}\gamma_2, \tag{55} \]

giving
\[ \frac{s_4 \beta^{(4)}_2}{\gamma_2} - \frac{j \beta^{(4)}_2}{\gamma_2} + \left\lfloor \frac{j \beta^{(4)}_2}{\gamma_2} \right\rfloor < s_3 < \frac{s_4 \beta^{(4)}_2}{\gamma_2} + t_{s_4}. \]

For \(3 \leq s_4 \leq 6\) we have \(0 < t_{s_4} < 1\) and the inequality for \(s_3\) may have two solutions \(s_3 = \left\lfloor \frac{s_4 \beta^{(4)}_2}{\gamma_2} \right\rfloor\) or \(s_3 = \left\lfloor \frac{s_4 \beta^{(4)}_2}{\gamma_2} \right\rfloor + 1\), giving rise to 4 more elements in \(B\).

For \(1 \leq s_4 \leq 2\) we have \(0 < t_{s_4} < 2\) and the inequality for \(s_3\) may have three solutions \(s_3 = \left\lfloor \frac{s_4 \beta^{(4)}_2}{\gamma_2} \right\rfloor\), \(s_3 = \left\lfloor \frac{s_4 \beta^{(4)}_2}{\gamma_2} \right\rfloor + 1\) or \(s_3 = \left\lfloor \frac{s_4 \beta^{(4)}_2}{\gamma_2} \right\rfloor + 2\). Below we shall show that \((s_2, \left\lfloor \frac{2 \beta^{(4)}_2}{\gamma_2} \right\rfloor + 2, 2)\) cannot occur when we claim (39). Therefore we define \(E = \{(s_2, 2, 1)\}\) and include this transfer in the list of possible substitutions for \(M(j)\) and \(K(j)\). Finally \(D\) has to be extended by \((s_2, 4, 0)\) since

\[ \epsilon_4 + \gamma_3 + j \beta^{(4)}_2 - \left\lfloor \frac{j \beta^{(4)}_2}{\gamma_2} \right\rfloor + 4 \gamma_2 \leq h + \delta \]

may be possible. Thus our new set of possible transferes consist of \(A, C\) and

\[
B^* = B \cup \left\{(s_2, s_3, s_4) \mid 3 \leq s_4 \leq 6, s_3 = \left\lfloor \frac{s_4 \beta^{(4)}_2}{\gamma_2} \right\rfloor + 1, s_2 = \left\lfloor \frac{3 \beta^{(3)}_1 + s_4 \beta^{(4)}_2}{\gamma_2} \right\rfloor \right\}
\]

\[
D^* = D \cup \{(s_2, 4, 0)\}
\]

\[
E = \{(s_2, 2, 1)\}
\]

Now we show why the use of \((s_2, \left\lfloor \frac{2 \beta^{(4)}_2}{\gamma_2} \right\rfloor + 2, 2)\) cannot occur for the extremal bases. We assume that this transfer has been used on \(M(j)\) or \(K(j)\). This gives us

\[
\epsilon_4 + 3 \gamma_3 + j \beta^{(4)}_2 - \left\lfloor \frac{j \beta^{(4)}_2}{\gamma_2} \right\rfloor \gamma_2 - 2 \beta^{(4)}_2 + \left\lfloor \frac{2 \beta^{(4)}_2}{\gamma_2} \right\rfloor \gamma_2 + 2 \gamma_2 \leq h + \delta.
\]

We now examine every \(j = 1, 2, \ldots, 6\). From (45) for \(t = 2\) we now that \(\epsilon_4 + 3 \gamma_3 + \frac{6}{5} \gamma_2 > h + \delta\). This inequality will be sufficient to exclude all the following cases.

a) If \(j = 1\) we get

\[
\epsilon_4 + 3 \gamma_3 - \beta^{(4)}_2 + \left\lfloor \frac{2 \beta^{(4)}_2}{\gamma_2} \right\rfloor \gamma_2 + 2 \gamma_2 \leq h + \delta.
\]

If \(\beta^{(4)}_2 < \frac{1}{2}\) we have \(\left\lfloor \frac{2 \beta^{(4)}_2}{\gamma_2} \right\rfloor = 0\) and (56) gives \(\epsilon_4 + 3 \gamma_3 + \frac{3}{2} \gamma_2 < h + \delta\), contradicting (45) for \(t = 2\).

If \(\beta^{(4)}_2 \geq \frac{1}{2}\) we have \(\left\lfloor \frac{2 \beta^{(4)}_2}{\gamma_2} \right\rfloor = 1\) and (56) gives \(\epsilon_4 + 3 \gamma_3 + 2 \gamma_2 < h + \delta\), contradicting (45) for \(t = 2\).

b) If \(j = 2\) we get

\[
\epsilon_4 + 3 \gamma_3 + 2 \gamma_2 \leq h + \delta
\]
contradicting (45) for $t = 2$.
c) If $j = 3$ we get

$$\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} - \left\lfloor \frac{3\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 + \left\lceil \frac{2\beta_2^{(4)}}{\gamma_2} \right\rceil \gamma_2 + 2\gamma_2 \leq h + \delta.$$ 

If $\frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{3}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = \left\lfloor \frac{3\beta_2^{(4)}}{\gamma_2} \right\rfloor = 0$ and (56) gives

$$\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + 2\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

If $\frac{1}{3} \leq \frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{2}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 0$ and $\left\lfloor \frac{3\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$, so (56) gives

$$\epsilon_4 + 3\gamma_3 + \frac{4}{3}\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

If $\frac{1}{2} \leq \frac{\beta_2^{(4)}}{\gamma_2} < \frac{2}{3}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$ and $\left\lfloor \frac{3\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$, so again (56) gives

$$\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + 2\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

Finally if $\frac{2}{3} \leq \frac{\beta_2^{(4)}}{\gamma_2} < 1$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$ and $\left\lfloor \frac{3\beta_2^{(4)}}{\gamma_2} \right\rfloor = 2$, so (56) gives

$$\epsilon_4 + 3\gamma_3 + \frac{5}{3}\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

d) If $j = 4$ we get

$$\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \left\lfloor \frac{4\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 + \left\lceil \frac{2\beta_2^{(4)}}{\gamma_2} \right\rceil \gamma_2 + 2\gamma_2 \leq h + \delta.$$ 

If $\frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{5}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = \left\lfloor \frac{4\beta_2^{(4)}}{\gamma_2} \right\rfloor = 0$ and (56) gives

$$\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} + 2\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

If $\frac{1}{5} \leq \frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{3}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 0$ and $\left\lfloor \frac{4\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$, so (56) gives

$$\epsilon_4 + 3\gamma_3 + \frac{3}{2}\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

If $\frac{1}{3} \leq \frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{2}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$ and $\left\lfloor \frac{4\beta_2^{(4)}}{\gamma_2} \right\rfloor = 2$, so again (56) gives

$$\epsilon_4 + 3\gamma_3 + 2\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

Finally if $\frac{1}{2} \leq \frac{\beta_2^{(4)}}{\gamma_2} < 1$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$ and $\left\lfloor \frac{4\beta_2^{(4)}}{\gamma_2} \right\rfloor = 3$, so (56) gives

$$\epsilon_4 + 3\gamma_3 + \frac{5}{3}\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

e) If $j = 5$ we get

$$\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} - \left\lfloor \frac{5\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 + \left\lceil \frac{2\beta_2^{(4)}}{\gamma_2} \right\rceil \gamma_2 + 2\gamma_2 \leq h + \delta.$$ 

If $\frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{5}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = \left\lfloor \frac{5\beta_2^{(4)}}{\gamma_2} \right\rfloor = 0$ and (56) gives

$$\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + 2\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

If $\frac{1}{5} \leq \frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{3}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 0$ and $\left\lfloor \frac{5\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$, so (56) gives

$$\epsilon_4 + 3\gamma_3 + \frac{8}{5}\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$.

If $\frac{1}{3} \leq \frac{\beta_2^{(4)}}{\gamma_2} < \frac{1}{2}$ we have $\left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1$ and $\left\lfloor \frac{5\beta_2^{(4)}}{\gamma_2} \right\rfloor = 2$, so (56) gives

$$\epsilon_4 + 3\gamma_3 + \frac{6}{5}\gamma_2 < h + \delta$$ 

contradicting (45) for $t = 2$. 
If \( \frac{1}{2} \leq \frac{\beta(4)}{\gamma_2} < \frac{3}{5} \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 1 \) and \( \left\lfloor \frac{3\beta(4)}{\gamma_2} \right\rfloor = 2 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + \frac{5}{3}\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

If \( \frac{3}{5} \leq \frac{\beta(4)}{\gamma_2} < \frac{4}{5} \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 1 \) and \( \left\lfloor \frac{3\beta(4)}{\gamma_2} \right\rfloor = 3 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + \frac{8}{5}\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

Finally if \( \frac{4}{5} \leq \frac{\beta(4)}{\gamma_2} < 1 \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 1 \) and \( \left\lfloor \frac{3\beta(4)}{\gamma_2} \right\rfloor = 4 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + \frac{7}{5}\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

Finally if \( j = 6 \) we get 
\[ \epsilon_4 + 3\gamma_3 + 4\beta_2^{(4)} - \left\lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 + \left\lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \right\rfloor \gamma_2 + 2\gamma_2 \leq h + \delta. \]

If \( \frac{\beta(4)}{\gamma_2} < \frac{1}{6} \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = \left\lfloor \frac{6\beta(4)}{\gamma_2} \right\rfloor = 0 \) and (56) gives 
\[ \epsilon_4 + 3\gamma_3 + 4\beta_2^{(4)} + 2\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

If \( \frac{1}{6} \leq \frac{\beta(4)}{\gamma_2} < \frac{1}{3} \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 0 \) and \( \left\lfloor \frac{6\beta(4)}{\gamma_2} \right\rfloor = 1 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + \frac{5}{3}\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

If \( \frac{1}{3} \leq \frac{\beta(4)}{\gamma_2} < \frac{1}{2} \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 0 \) and \( \left\lfloor \frac{6\beta(4)}{\gamma_2} \right\rfloor = 2 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + \frac{4}{3}\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

If \( \frac{1}{2} \leq \frac{\beta(4)}{\gamma_2} < \frac{2}{3} \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 1 \) and \( \left\lfloor \frac{6\beta(4)}{\gamma_2} \right\rfloor = 3 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

If \( \frac{2}{3} \leq \frac{\beta(4)}{\gamma_2} < \frac{5}{6} \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 1 \) and \( \left\lfloor \frac{6\beta(4)}{\gamma_2} \right\rfloor = 4 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + \frac{5}{3}\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

Finally if \( \frac{5}{6} \leq \frac{\beta(4)}{\gamma_2} < 1 \) we have \( \left\lfloor \frac{2\beta(4)}{\gamma_2} \right\rfloor = 1 \) and \( \left\lfloor \frac{6\beta(4)}{\gamma_2} \right\rfloor = 5 \), so (56) gives 
\[ \epsilon_4 + 3\gamma_3 + \frac{4}{3}\gamma_2 < h + \delta \] contradicting (45) for \( t = 2 \).

We now return to the unsolved case number 8 from interval \( I_2 \), where two possible inequality systems "survived" (see the end of the previous section). Both systems contain a line where \([0, 5], 0, 5\) is used giving 
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - 5\beta_2^{(4)} \leq h + \delta. \] (57)

This present example will show how to exploit the information coming from the key numbers of higher order in this case \(M(5)\) and \(K(5)\). It turns out that only very few transfers from \(A \cup B' \cup C \cup D' \cup E\) are possible in \(K(5)\). Of course \(M(5)\) is chosen in such a way that the transfer \([0, 5], 0, 5\) in \(A\) is impossible in the whole interval \(K(5)\). It would give a negative coefficient in the second place. If we use \((s_2, s_3, 6) \in A\) for some number from \(K(5)\) we would get 
\[ \epsilon_4 + 7\gamma_3 + \gamma_2 - \beta_2^{(4)} \leq h + \delta. \]

Since \(\beta_2^{(4)} \leq \gamma_2/5\) this gives \(\epsilon_4 + 7\gamma_3 + 4\gamma_2/5 \leq h + \delta\), contradicting (45) for \(t = 6\). All transfers \((s_2, s_3, s_4) \in A\) for \(s_4 \leq 4\) are possible.
1.) If \((s_2, s_3, 0) \in D^*, s_3 \geq 1\) is used we have
\[
\epsilon_4 + \gamma_3 + 5\beta_2^{(4)} + s_3\gamma_2 \leq h + \delta.
\]
Averaging with (57) gives \(\epsilon_4 + 7\gamma_3/2 + \gamma_2 \leq h + \delta\), giving a coefficient bound below 2 according to (41) and (43).

2.) If \((s_2, 2, 1) \in E\) is used we have
\[
\epsilon_4 + 2\gamma_3 + 4\beta_2^{(4)} + 2\gamma_2 \leq h + \delta.
\]
contradicting (45) for \(t = 1\) since \(\beta_2^{(4)} \geq \gamma_2/6\).

3.) If \((s_2, s_3, s_4) \in B^*\) is used we have \(s_3 = 1 + \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor\) and thus
\[
\epsilon_4 + (1 + s_4)\gamma_3 + 5\beta_2^{(4)} - s_4\beta_2^{(4)} + (1 + \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor)\gamma_2 \leq h + \delta.
\]
Averaging with (57) gives
\[
\epsilon_4 + (7 + s_4)\gamma_3/2 + \gamma_2 - s_4\beta_2^{(4)}/2 + \left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor\gamma_2/2 \leq h + \delta,
\]
contradicting (45) for \(s_4 \leq 5\). For \(s_4 = 6\) we have \(\left\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \right\rfloor = 1\) and we even get
\[
\epsilon_4 + 13\gamma_3/2 + 3\gamma_2/2 - 3\beta_2^{(4)} \leq h + \delta.
\]
also contradicting (45) for \(t = 5\).
Thus the only possible transfers are \((s_2, s_3, s_4) \in A\) for \(s_4 \leq 4\) in addition to the regular one and we can write down the inequality system for \(K(5)\). We consider (53). The ordering of the transfers \((s_2, s_3, s_4) \in A\) for \(s_4 \leq 4\) has to be the same in the list for \(M(5)\) as in the main list because of the decreasing order of the reductions of the constant terms. Therefore we get.

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + 4\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 5\beta_1^{(4)} + 4\beta_2^{(4)} - 3\gamma_1 & \leq h + \delta.
\end{align*}
\]

It is easy to see that the first 4 lines do not give additional information compared to (53). But the last line is new. So we would expect that this line enters the optimal weighting of the inequality systems and it does. If we use the weight distribution 0,0,0,3,1,0,0,0,0,0,1, where we put (53) first this yields
\[
\epsilon_4 + 22\gamma_3/5 + 8\gamma_2/25 + \gamma_1/5 \leq h + \delta.
\]
giving a coefficient bound 2.004 enough to exclude the whole case.

If we start with (54), we have to chose the same ordering for the transfers $(s_2, s_3, s_4) \in A$ for $s_4 \leq 4$ for $M(5)$ as in the main list because of the decreasing order of the reductions of the constant terms. Now we get:

\[
\begin{align*}
\epsilon_4 + 5\gamma_3 + 5\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + 4\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 5\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta,
\end{align*}
\]

Again it is easy to see that the last 4 lines do not give any additional information compared to (54). But the first line is new. So we would expect that this line enters the optimal weighting of the inequality systems and it does. If we use the weight distribution 0,0,2,0,0,1,0,0,0,0 where we put (54) first this yields

\[
\epsilon_4 + 19\gamma_3/5 + 11\gamma_2/25 + \gamma_1/5 \leq h + \delta,
\]

giving a coefficient bound 2.005 enough to exclude the whole case. Thus we have finished interval $I_2$. The extremal bases cannot be found there.

The computer programme presented in chapter 13 goes through the actual cases and generates the corresponding inequality systems for the main list and for key numbers of higher order. Sometimes it is not enough to exploit a single key number of higher order and several levels of key numbers have to be included.

## 10 Combination of Lists

In the chapter about the average method we introduced the main list consisting of the coefficient inequalities for the representations of the key numbers belonging to the main list. Since we first consider the situation where the regular representation stands at the end of the list we have $(s_2^{(l)}, s_3^{(l)}, s_4^{(l)}) = (0, 0, 0)$. Here we have $\kappa_l = 0$.

\[
\begin{align*}
\epsilon_4 + (s_4^{(1)} + 1)\gamma_3 + (s_3^{(1)} + 1)\gamma_2 - s_4^{(1)}\beta_2^{(4)} + \gamma_1 - \kappa_1 & \leq h + \delta \\
\epsilon_4 + (s_4^{(2)} + 1)\gamma_3 + (s_3^{(2)} + 1)\gamma_2 - s_4^{(2)}\beta_2^{(4)} + \kappa_1 - \kappa_2 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + \kappa_{l-1} & \leq h + \delta
\end{align*}
\]

In order to avoid too much unnecessary notation we introduce the following magnitudes for the reduction of the second term caused by a transfer
Here the coefficients $x_i$ since $z_i$ lines in the list for $M$ and their only candidates for $\sigma_i > 0$ is the largest reduction of the second term occurring in the main list. We now focus on the corresponding key number

$$M = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\sigma_i - 1)a_2 + (\gamma_1 - 1),$$

a number that does not allow the transfer $(s_2^{(i)}, s_3^{(i)}, s_4^{(i)})$. No other transfer can have the same maximal reduction $\sigma_i$ since the elements of $A$ have different reductions of the second term and they are the only candidates for $(s_2^{(i)}, s_3^{(i)}, s_4^{(i)})$.

Of course the elements $n \in [M - \gamma_2 + 1, M]$ must have an $h$-representation. These representations of course are not so different from those in the main list but there are some differences. The following observation is easy to see. If $n = x_4a_4 + x_3a_3 + x_2a_2 + x_1$ is a minimal representation of $n$, and $0 \leq z_i \leq x_i$ for $i = 1, 2, ..., 4$ then $m = z_4a_4 + z_3a_3 + z_2a_2 + z_1$ is a minimal representation of $m$. Otherwise there are another representation $m = z'_4a_4 + z'_3a_3 + z'_2a_2 + z'_1$ with lower coefficient sum. But then $n = n - m + m = (x_4 - z_4 + z'_4)a_4 + (x_3 - z_3 + z'_3)a_3 + (x_2 - z_2 + z'_2)a_2 + (x_1 - z_1 + z'_1)$. Here the coefficients $x_i, z_i$ are nonnegative since $z_i \leq x_i$ and the coefficient sum $(x_4 - z_4 + z'_4) + (x_3 - z_3 + z'_3) + (x_2 - z_2 + z'_2) + (x_1 - z_1 + z'_1) < x_4 + x_3 + x_2 + x_1$ since $z'_4 + z'_3 + z'_2 + z'_1 < z_4 + z_3 + z_2 + z_1$, a contradiction. That means that the first $i - 1$ lines in the list for $M$ must contain the same transferes as the corresponding lines of the main list. Thus we have

$$\epsilon_4 + (s_4^{(i)} + 1)\gamma_3 + \sigma_i - \sigma_1 + \gamma_1 - \kappa_1 \leq h + \delta$$

$$\epsilon_4 + (s_4^{(2)} + 1)\gamma_3 + \sigma_i - \sigma_2 + \kappa_1 - \kappa_2 \leq h + \delta$$

$$\epsilon_4 + \gamma_3 + \gamma_2 + \kappa_{i-1} \leq h + \delta$$
\[ \epsilon_4 + (s_4^{(i-1)} + 1)\gamma_3 + \sigma_i - \sigma_{i-1} + \kappa_{i-2} - \kappa_{i-1} \leq h + \delta \]

None of these lines contains new information since the coefficient sums here are smaller than the corresponding ones from the main list since of course \( \gamma_2 > \sigma_i \).

We have now arrived at the following key number

\[ P_1 = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\sigma_i - 1)a_2 + (\kappa_{i-1} - 1), \]

where we cannot use the transfer \((s_2^{(i)}, s_3^{(i)}, s_4^{(i)})\). But the transfere \((s_2^{(i+1)}, s_3^{(i+1)}, s_4^{(i+1)})\) is possible, since \( \sigma_i > \sigma_{i+1} \) and \( \kappa_{i-1} > \kappa_{i+1} \). If this transfer already gives us the optimal representation of \( P_1 \) we get

\[ \epsilon_4 + (s_4^{(i+1)} + 1)\gamma_3 + \sigma_i - \sigma_{i+1} + \kappa_{i-1} - \kappa_{i+1} \leq h + \delta. \]

The remainder of the list for key number \( M \) is now a copy of the main list where the second coefficient is reduced from \( \gamma_2 \) to \( \sigma_i \). The same transfers as in the main list are used here, so there is no new information to get from this remainder but the inequality for the coefficients of \( P_1 \) is new.

If there is another transfere \( S'_1 \) from \( A, B^*, C, D^* \) or \( E \) suitable for \( P_1 \) and with a better gain than \( S_{i+1} \), we get

\[ \epsilon_4 + (s_4^{(i+1)} + 1)\gamma_3 + \sigma_i - \sigma_{i+1} + \kappa_{i-1} - \kappa_{i+1} \leq h + \delta. \]

We also see that \( \kappa'_1 > \kappa_i \). Otherwise the transfere \( S'_1 \) would be possible in line \( i + 1 \) in the main list with a greater gain than \( S_{i+1} \), which of course is not possible. We continue by considering the key number

\[ P_2 = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\sigma_i - 1)a_2 + (\kappa'_1 - 1). \]

Again we know that \( S_{i+1} \) is possible and if it is optimal we get

\[ \epsilon_4 + (s_4^{(i+1)} + 1)\gamma_3 + \sigma_i - \sigma_{i+1} + \kappa'_1 - \kappa_{i+1} \leq h + \delta. \]

and the remainder of the list again follows the main list without giving additional information. Now if there is another transfere \( S'_2 \) from \( A, B^*, C, D^* \) or \( E \) suitable for \( P_2 \) and with a better gain than \( S_{i+1} \), we get

\[ \epsilon_4 + (s_4^{(i+2)} + 1)\gamma_3 + \sigma_i - \sigma_{i+1} + \kappa'_1 - \kappa'_2 \leq h + \delta. \]

Still we have \( \kappa'_2 > \kappa_i \) by the same reason as above. In the same manner we can continue until for some \( P_{L-1} \) the transfere \( S_{i+1} \) is used and we get the following collection of \( L \) inequalities:
\[\epsilon_4 + (s^{(1)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_1 + \gamma_1 - \kappa_1 \leq h + \delta \]
\[\epsilon_4 + (s^{(2)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_2 + \kappa_1 - \kappa_2 \leq h + \delta \]
\[\ldots \]
\[\epsilon_4 + (s^{(i+1)}_4 + 1)\gamma_3 + \sigma_i - \sigma_{i+1} + \kappa'_{L-1} - \kappa_{i+1} \leq h + \delta \]

Here we have for the reductions of the constant term
\[\kappa_{i-1} > \kappa'_1 > \kappa'_2 > \ldots > \kappa'_{L-1} > \kappa_i \]

and
\[G(S^{(i)}) > G(S^{(1)}) > G(S^{(2)}) > \ldots > G(S^{(L-1)}) > G(S^{(L)}) = G(S^{(i+1)}) \]

for the gains of the corresponding transfers.

Below only the transfers giving new information are gathered:

\[\epsilon_4 + (s^{(1)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_1 + \gamma_1 - \kappa_1 \leq h + \delta \] (weight 1)
\[\epsilon_4 + (s^{(2)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_2 + \kappa_1 - \kappa_2 \leq h + \delta \] (weight 1)
\[\ldots \]
\[\epsilon_4 + (s^{(i-1)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_{i-1} + \kappa_{i-2} - \kappa_{i-1} \leq h + \delta \] (weight 1)
\[\epsilon_4 + (s^{(i)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_i + \kappa_{i-1} - \kappa_i \leq h + \delta \] (weight 1 - \(x\))
\[\epsilon_4 + (s^{(i+1)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_{i+1} + \kappa_i - \kappa_{i+1} \leq h + \delta \] (weight 1 - \(x\))
\[\epsilon_4 + (s^{(i+2)}_4 + 1)\gamma_3 + \gamma_2 - \sigma_{i+2} + \kappa_{i+1} - \kappa_{i+2} \leq h + \delta \] (weight 1)
\[\ldots \]
\[\epsilon_4 + (s^{(L-1)}_4 + 1)\gamma_3 + \gamma_2 + \kappa_{L-1} \leq h + \delta \] (weight 1)
\[\epsilon_4 + (s^{(L)}_4 + 1)\gamma_3 + \sigma_i - \sigma'_{i+1} + \kappa'_{L-1} - \kappa_i \leq h + \delta \] (weight \(x\))
\[\epsilon_4 + (s^{(2)}_4 + 1)\gamma_3 + \sigma_i - \sigma'_{i+1} + \kappa'_{L-1} - \kappa_i \leq h + \delta \] (weight \(x\))
\[\ldots \]

For the sum \(T\) of the weights we have \(T = l + (L - 2)x\). Averaging all the inequalities using the proposed positive weights \(0 \leq x \leq 1\) this yields:

\[\epsilon_4 + \left(1 + \frac{\sum_{k=1}^{l} s^{(k)}_4 - x s^{(i)}_4 + x \sum_{k=1}^{L-1} s^{(k)}_4}{T} \right) \gamma_3 + \frac{l \gamma_2 - \sum_{k=1}^{l} \sigma_k + x ((L + 1) \sigma_i - 2 \gamma_2 - \sum_{k=1}^{L-1} \sigma'_{k}) + \gamma_1}{T} \leq h + \delta \] (58)
In this way we produce a new inequality of the type (41) which can help us to
exclude the case. In the computer programme presented in chapter 13 this fact
is exploited.

10.1 The additional contribution argument

If we put \( x = 1 \) in the results of the previous section we get

\[
\epsilon_4 + \left( 1 + \frac{\sum_{k=1}^{l} s_4^{(k)} - s_4^{(i)} + \sum_{k=1}^{L-1} s_4^{(k)}}{l - 2 + L} \right) \gamma_3 + \\
\frac{(l - 2) \gamma_2 - \sum_{k=1}^{l} \sigma_k + ((L + 1) \sigma_i - \sum_{k=1}^{L-1} \sigma_k')}{l - 2 + L} \gamma_1 + \frac{1}{l - 2 + L} \leq h + \delta
\]

This inequality can be seen as a combination of the main list and the list for
\( M \) in the following way. First we gather the lines 1, 2, ..., \( i - 1 \) from the main list.
Then we gather the \( L \) lines from the list for \( M \) and in the end we gather the
lines \( i + 2, i + 3, \ldots, l \) from the main list again as seen below.

\[
\epsilon_4 + (s_4^{(1)} + 1) \gamma_3 + \gamma_2 - \sigma_1 + \gamma_1 - \kappa_1 \leq h + \delta \\
\epsilon_4 + (s_4^{(2)} + 1) \gamma_3 + \gamma_2 - \sigma_2 + \kappa_1 - \kappa_2 \leq h + \delta \\
\ldots \\
\epsilon_4 + (s_4^{(i-1)} + 1) \gamma_3 + \gamma_2 - \sigma_{i-1} + \kappa_{i-1} - \kappa_{i-1} \leq h + \delta \\
\epsilon_4 + (s_4^{(i+1)} + 1) \gamma_3 + \gamma_2 - \sigma_{i+1} + \kappa_{L-1} - \kappa_{i+1} \leq h + \delta \\
\epsilon_4 + (s_4^{(i+2)} + 1) \gamma_3 + \gamma_2 - \sigma_{i+2} + \kappa_{i+2} - \kappa_{i+2} \leq h + \delta \\
\ldots \\
\epsilon_4 + \gamma_3 + \gamma_2 + \kappa_{l-1} \leq h + \delta
\]

This can be seen as the list for \( M \) where \( l - 2 \) of the lines have got the
additional contribution \( \gamma_2 - \sigma_i \). So if we know the transfers in the main list and
in the list for \( M \) we can calculate the average inequality with the additional
contribution without knowing the ordering of the transfers. This inequality
usually gives a much lower coefficient than the average inequality for \( M \) itself.
This argument is called the additional contribution argument and is used many
times below.
Below one can see the average inequality for the list for \( M \).
\[ \epsilon_4 + \left( 1 + \frac{\sum_{k=1}^{l} s_4^{(k)} - s_4^{(i)} + \sum_{k=1}^{L-1} s_4^{(k)}}{l - 2 + L} \right) \gamma_3 + \right) \gamma_3 + \frac{(l - 2)\sigma_i - \sum_{k=1}^{l} \sigma_k + ((L + 1)\sigma_i - \sum_{k=1}^{L-1} \sigma_k')}{l - 2 + L} \leq h + \delta \]

In this form we easily see the amount of the additional contribution \((l - 2)(\gamma_2 - \sigma_i)\).

The same argument can be used for key numbers of higher order, too. We start with the list for some key number

\[ H = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\sigma - 1)a_2 + (\gamma_1 - 1). \]

This list then reads

\[
\begin{align*}
\epsilon_4 + (s_4^{(1)} + 1)\gamma_3 + \sigma - \sigma_1 + \gamma_1 - \kappa_1 & \leq h + \delta \\
\epsilon_4 + (s_4^{(2)} + 1)\gamma_3 + \sigma - \sigma_2 + \kappa_1 - \kappa_2 & \leq h + \delta \\
\vdots & \vdots \\
\epsilon_4 + \gamma_3 + \sigma + \kappa_{l-1} & \leq h + \delta
\end{align*}
\]

If \(\sigma'\) is the largest occurring reduction of the second coefficient in that list (line \(i\)), we consider the key number

\[ J = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\sigma' - 1)a_2 + (\kappa_{i-1} - 1). \]

and find in the same way as before the new inequalities

\[
\begin{align*}
\epsilon_4 + (s_4^{(i+1)} + 1)\gamma_3 + \sigma' - \sigma_i' + \kappa_{i-1} - \kappa_{i-1}' & \leq h + \delta \\
\epsilon_4 + (s_4^{(2)} + 1)\gamma_3 + \sigma' - \sigma_i' + \kappa_i' - \kappa_2' & \leq h + \delta \\
\vdots & \vdots \\
\epsilon_4 + (s_4^{(i+1)} + 1)\gamma_3 + \sigma' - \sigma_{i+1} + \kappa_{L-1}' - \kappa_{i+1} & \leq h + \delta
\end{align*}
\]

Combining the two systems in the same way as before also here gives an additional contribution. In this case the contribution is \((l - 2)(\sigma - \sigma')\). The new average inequality usually gives much better results than the inequality without additional contribution. This argument is often used in the chapter about \(s > 0\).
11 Exclusion of certain transfers

In the section about the average method we also showed that \( r_6 = 0 \) in the main list in the intervals \( I_2, I_3, I_5, I_7, I_9, I_{10} \). This gave a huge reduction in the number of cases to consider. We need not consider interval \( I_1 \) since we already showed that the extremal bases cannot be there. We now show how to exclude \( r_6 = 1 \) also from the other intervals, i.e. we want to exclude the use of \( ([6, \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor], 6) \in A \) from the main list. The reduction in the number of cases is in fact essential for the computer programme (see, chapter 13) to be able to do the job in reasonable time.

Now we assume that \( ([\lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor], 6), [\lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor], 6) \) has been used in the main list giving the inequality

\[
\epsilon_4 + 7\gamma_3 + (1 + \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor)\gamma_2 - 6\beta_2^{(4)} \leq h + \delta, \tag{59}
\]

where we left out the constant term. We will show that this assumption leads to a contradiction.

11.1 The intervals \( I_4 \) and \( I_6 \)

In interval \( I_4 \) and \( I_6 \) we can use similar arguments, so we treat them here together. In interval \( I_4 \) we have \( \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor = 1 \) and in interval \( I_6 \) we have \( \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor = 2 \). We consider a new key number

\[
M(6) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (6\beta_2^{(4)} - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor)\gamma_2 - 6\beta_2^{(4)} a_2 + (\gamma_1 - 1)
\]

and try to find a representation for \( M(6) \) and all the numbers in the interval \( K(6) = [M(6) - \gamma_1 + 1, M(6)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(6) \). It turns out that only very few transfers are possible.

1.) If \((s_2, s_3, 0) \in D^*, s_3 \geq 1 \) is used we have

\[
\epsilon_4 + \gamma_3 + 6\beta_2^{(4)} - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor \gamma_2 + s_3\gamma_2 \leq h + \delta.
\]

Averaging with (59) gives \( \epsilon_4 + 4\gamma_3 + \gamma_2 \leq h + \delta \), contradicting (45) for \( t = 3 \).

2.) If \((\lfloor 2, 1 \rfloor, 2, 1) \in E \) is used we have

\[
\epsilon_4 + 2\gamma_3 + 6\beta_2^{(4)} - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rfloor \gamma_2 - \beta_2^{(4)} + 2\gamma_2 \leq h + \delta.
\]

Averaging with (59) gives \( \epsilon_4 + 9\gamma_3/2 + \gamma_2 + (\gamma_2 - \beta_2^{(4)})/2 \leq h + \delta \), again contradicting (45) for \( i = 3 \).
3.) The minimal representation of $M(6)$ cannot be the regular one. Otherwise we would have
\[
\epsilon_4 + \gamma_3 + 6\beta_2^{(4)} - \left[ \frac{6\beta_2^{(4)}}{\gamma_2} \right] \gamma_2 + \gamma_1 \leq h + \delta.
\]
Averaging with (59) gives $\epsilon_4 + 4\gamma_3 + \gamma_2 / 2 + \gamma_1 / 2 \leq h + \delta$, yielding a coefficient bound 1 far below 2.008 according to (41).

4.) If $(s_2, s_3, s_4) \in B^*$ is used we have $s_3 = 1 + \left[ \frac{s_4\beta_2^{(4)}}{\gamma_2} \right]$ and thus
\[
\epsilon_4 + (1 + s_4)\gamma_3 + 6\beta_2^{(4)} - \left[ \frac{6\beta_2^{(4)}}{\gamma_2} \right] \gamma_2 - s_4\beta_2^{(4)} + (1 + \left[ \frac{s_4\beta_2^{(4)}}{\gamma_2} \right]) \gamma_2 \leq h + \delta.
\]
Again averaging with (59) gives
\[
\epsilon_4 + (4 + \frac{s_4}{2})\gamma_3 + \frac{\gamma_2}{2} + \left[ \frac{s_4\beta_2^{(4)}}{\gamma_2} \right] + (1 + \frac{s_4\beta_2^{(4)}}{\gamma_2}) \gamma_2 - s_4\beta_2^{(4)} \leq h + \delta.
\]
which contradicts (45) for $t = 4$ if $s_4 > 1$ since $\left[ \frac{s_4\beta_2^{(4)}}{\gamma_2} \right] + (1 + \frac{s_4\beta_2^{(4)}}{\gamma_2}) \geq 0$. In interval $I_4$ and $I_6$ we have $\beta_2^{(4)} \leq \gamma_2 / 2$ and the average-inequality for $s_4 = 1$ reads
\[
\epsilon_4 + \frac{9}{2}\gamma_3 + \frac{3\gamma_2}{4} \leq \epsilon_4 + (4 + \frac{1}{2})\gamma_3 + \frac{\gamma_2}{2} + \frac{\gamma_2 - \beta_2^{(4)}}{2} \leq h + \delta.
\]
yielding a coefficient bound $1.80 < 2.008$ according to (41) and (43).

Of course the numbers in $K(6)$ do not allow the transfer $([6\beta_2^{(4)} \gamma_2], [\frac{6\beta_2^{(4)}}{\gamma_2}], 6) \in A$, because otherwise the second coefficient of the representation of $n \in K(6)$ would be negative. If $(s_2, s_3, 5) \in A$ was used we get
\[
\epsilon_4 + 6\gamma_3 + \beta_2^{(4)} + (s_3 - \left[ \frac{6\beta_2^{(4)}}{\gamma_2} \right]) \gamma_2 \leq h + \delta.
\]
In interval $I_6$ we have $\left[ \frac{6\beta_2^{(4)}}{\gamma_2} \right] = 2 = s_3 = \left[ \frac{5\beta_2^{(4)}}{\gamma_2} \right]$ since $\gamma_2 / 2 \geq \beta_2^{(4)} \geq 2\gamma_2 / 5$. Thus the inequality contradicts (45) for $t = 5$.

In interval $I_4$ we have $\left[ \frac{6\beta_2^{(4)}}{\gamma_2} \right] = 1 = s_3 = \left[ \frac{5\beta_2^{(4)}}{\gamma_2} \right]$. Together with (59) and the weight distribution 6,1 we get
\[
\epsilon_4 + 43\gamma_3 / 7 + 2\gamma_2 / 7 \leq h + \delta,
\]
giving a coefficient below $1.97 < 2.008$ by (41) and (43).

If $(s_2, s_3, 4) \in A$ was used we get
\[
\epsilon_4 + 5\gamma_3 + 2\beta_2^{(4)} + (s_3 - \left[ \frac{6\beta_2^{(4)}}{\gamma_2} \right]) \gamma_2 \leq h + \delta.
\]
In interval $I_4$ we have $\lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rceil = 1 = s_3 = \lfloor \frac{4\beta_2^{(4)}}{\gamma_2} \rceil$ and in interval $I_6$ we have $\lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rceil = 2 = s_3 = \lfloor \frac{4\beta_2^{(4)}}{\gamma_2} \rceil$. In both intervals we have $\beta_2^{(4)} \geq \gamma_2/4$. Thus the inequality contradicts (45) for $t = 4$.

Now the transfer $(s_2, s_3, 3) \in A$ is impossible in $I_4$. It would produce a negative coefficient in the representation of $n \in K(6)$ since

$$6\beta_2^{(4)} - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rceil \gamma_2 < 3\beta_2^{(4)} - \lfloor \frac{3\beta_2^{(4)}}{\gamma_2} \rceil \gamma_2 = 3\beta_2^{(4)}$$

and the transfer $(s_2, s_3, 2) \in A$ is impossible in $I_6$ since it would produce a negative coefficient in the representation of $n \in K(6)$ since $6\beta_2^{(4)} - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rceil \gamma_2 < 2\beta_2^{(4)} - \lfloor \frac{2\beta_2^{(4)}}{\gamma_2} \rceil \gamma_2 = 2\beta_2^{(4)}$.

Table 5. The interval $I_4$

<table>
<thead>
<tr>
<th>$K(6)$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_5$</th>
<th>$r_6$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_4$</th>
<th>$q_5$</th>
<th>$q_6$</th>
<th>$q$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_2$ - coef</td>
<td>M</td>
<td>M</td>
<td>-</td>
<td>-</td>
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</tr>
<tr>
<td>$L_n(A_2(h))$</td>
<td>$\leq$</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<td>2</td>
</tr>
</tbody>
</table>

Table 6. The interval $I_6$

<table>
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<tr>
<th>$K(6)$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_5$</th>
<th>$r_6$</th>
<th>$q_1$</th>
<th>$q_2$</th>
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<th>$q_5$</th>
<th>$q_6$</th>
<th>$q$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_2$ - coef</td>
<td>M</td>
<td>-</td>
<td>M</td>
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</tr>
<tr>
<td>$L_n(A_2(h))$</td>
<td>$\leq$</td>
<td>2</td>
<td>2</td>
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</tbody>
</table>

These tables are to be read in the following way: The symbol "M" in the line for the $a_2$ - coef means that the corresponding transfer may be possible whilst the symbol "-" stands for a negative coefficient in front of $a_2$. In the line for $L_n(A_2(h))$ the number 2 means that the coefficient for the $h$-range is $\leq 2$ and the corresponding transfer cannot be used. Thus we are left with the following possibilities:

In interval $I_4$ the following transfers are possible: $([0, 2], 0, 2), (1, 0, 0)$ and $(0, 0, 0)$. In interval $I_6$ the following transfers are possible: $([0, 3], 0, 3), (1, 0, 0)$ and $(0, 0, 0)$.

First we look at $(1, 0, 0)$ together with the regular representation at the end of the list since this situation is in common for both intervals.

$$\epsilon_4 + 2\gamma_3 + 5\beta_2^{(4)} - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rceil \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta$$

$$\epsilon_4 + \gamma_3 + 6\beta_2^{(4)} - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rceil \gamma_2 + \kappa_1 \leq h + \delta.$$ 

Giving $\epsilon_4 + 3\gamma_3/2 + 11\beta_2^{(4)}/2 - \lfloor \frac{6\beta_2^{(4)}}{\gamma_2} \rceil \gamma_2 + \gamma_1/2 \leq h + \delta$. Averaging with (59) gives

$$\epsilon_4 + 17\gamma_3/4 + \gamma_2/2 - \beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta.$$
Since $\beta_2^{(4)} \leq \gamma_2/2$ we get $\epsilon_4 + 17\gamma_3/4 + 3\gamma_2/8 + \gamma_1/4 \leq h + \delta$ yielding a coefficient $1.8 < 2.008$ according to (41) and (43). Therefore $(1, 0, 0)$ together with $(0, 0, 0)$ cannot be used for the numbers in $K(6)$ neither for $I_4$ nor $I_5$.

Now we have two possibilities left in either interval. We start with $I_4$, where $\beta_2^{(4)} = 1$.

First we look at $([0, 2], 0, 2) \in A$ together with the regular representation at the end of the list.

\[
\epsilon_4 + 3\gamma_3 + 4\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 6\beta_2^{(4)} - \gamma_2 + \kappa_1 \leq h + \delta.
\]

Averaging with (59) with equal weights gives

\[
\epsilon_4 + 11\gamma_3/3 + 4\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta.
\]

Since $\beta_2^{(4)} \geq \gamma_2/4$ we get $\epsilon_4 + 11\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta$ yielding a coefficient $1.94 < 2.008$ according to (41) and (43). Therefore $([0, 2], 0, 2)$ together with $(0, 0, 0)$ cannot be used for the numbers in $K(6)$ for $I_4$.

We could also have the situation that all three transfers $([0, 2], 0, 2), (1, 0, 0)$ and $(0, 0, 0)$ are used in $K(6)$. Here we have two possible orderings for the transfers

\[
\epsilon_4 + 3\gamma_3 + 4\beta_2^{(4)} - \gamma_2 + \gamma_1 - 2\beta_1^{(4)} + [2, 0]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 2\gamma_3 + 5\beta_2^{(4)} - \gamma_2 + 2\beta_1^{(4)} - [2, 0]\gamma_1 - \beta_1^{(4)} \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 6\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} \leq h + \delta.
\]

Because of the constant term $2\beta_1^{(4)} - [2, 0]\gamma_1 - \beta_1^{(4)} = \beta_1^{(4)} - [2, 0]\gamma_1$ in line 2 we can tell that $[2, 0] = 0$. Now we can use the weights 1, 0, 2 and 4 for (59) yielding

\[
\epsilon_4 + 33\gamma_3/7 + 5\gamma_2/7 - 8\beta_2^{(4)}/7 + \gamma_1/7 \leq h + \delta.
\]

Since $\beta_2^{(4)} \leq \gamma_2/3$ we get $\epsilon_4 + 33\gamma_3/7 + \gamma_2/3 + \gamma_1/7 \leq h + \delta$ yielding a coefficient $2.005 < 2.008$ according to (41) and (43).

We now consider the other ordering of the transfers.

\[
\epsilon_4 + 2\gamma_3 + 5\beta_2^{(4)} - \gamma_2 + \gamma_1 - \beta_1^{(4)} \leq h + \delta
\]
\[
\epsilon_4 + 3\gamma_3 + 4\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} - 2\beta_1^{(4)} - [2, 0]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 6\beta_2^{(4)} - \gamma_2 + 2\beta_1^{(4)} - [2, 0]\gamma_1 \leq h + \delta.
\]

Because of the constant term $\beta_1^{(4)} - 2\beta_1^{(4)} + [2, 0]\gamma_1 = [2, 0]\gamma_1 - \beta_1^{(4)}$ in line 2 we can tell that $[2, 0] = 1$. Now we can use the weights 2, 0, 1 and 4 for (59) yielding the same result as above. Therefore $([0, 2], 0, 2), (1, 0, 0)$ together with $(0, 0, 0)$
cannot be used for the numbers in $K(6)$ for $I_4$. Thus we have excluded $r_6 = 1$ from $I_4$.

There are also two possibilities left in $I_6$. Here $\lfloor \frac{6\beta(4)}{\gamma_2} \rfloor = 2$.

We first look at $([1, 3], 1, 3)$ together with the regular representation at the end of the list.

$$\epsilon_4 + 4\gamma_3 + 3\beta_2(4) - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta$$
$$\epsilon_4 + \gamma_3 + 6\beta_2(4) - 2\gamma_2 + \kappa_1 \leq h + \delta.$$ 

Averaging with (59) with equal weights gives

$$\epsilon_4 + 4\gamma_3 + \beta_2(4) + \gamma_1/3 \leq h + \delta.$$ 

Since $\beta_2(4) \geq 2\gamma_2/5$ we get $\epsilon_4 + 4\gamma_3 + 2\gamma_2/5 + \gamma_1/3 \leq h + \delta$ yielding the coefficient $15/8 < 2.008$ according to (41). Therefore $([1, 3], 1, 3)$ together with $(0, 0, 0)$ cannot be used for the numbers in $K(6)$ for $I_6$.

We could also have the situation that all three transfers $([1, 3], 1, 3)$, $(1, 0, 0)$ and $(0, 0, 0)$ are used in $K(6)$. We then have

$$\epsilon_4 + 4\gamma_3 + 3\beta_2(4) - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta$$
$$\epsilon_4 + 2\gamma_3 + 5\beta_2(4) - 2\gamma_2 + \kappa_1 - \kappa_2 \leq h + \delta$$
$$\epsilon_4 + \gamma_3 + 6\beta_2(4) - 2\gamma_2 + \kappa_2 \leq h + \delta.$$ 

Averaging with (59) using the weight distribution 1,1 and 3 for (59) we get

$$\epsilon_4 + 28\gamma_3/6 + 4\gamma_2/6 - 4\beta_2(4)/6 + \gamma_1/6 \leq h + \delta.$$ 

Since $\beta_2(4) \leq \gamma_2/2$ we get $\epsilon_4 + 14\gamma_3/3 + \gamma_2/3 + \gamma_1/6 \leq h + \delta$ yielding a coefficient below 2 according to (41) and (43). The other ordering of the transfers leaves us the same average inequality. Therefore $([1, 3], 1, 3)$, $(1, 0, 0)$ together with $(0, 0, 0)$ cannot be used for the numbers in $K(6)$ for $I_6$. Thus we have excluded $r_6 = 1$ from $I_6$, too.

### 11.2 Interval $I_8$

Our next goal is to exclude $r_6 = 1$ from the main list of $I_8$, where we have $\lfloor \frac{6\beta(4)}{\gamma_2} \rfloor = 3$. Here we study the key number

$$M(4) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (4\beta_2(4) - 2\gamma_2 - 1)a_2 + (\gamma_1 - 1)$$
and try to find a representation for $M(4)$ and all the numbers in the inverall $K(4) = [M(4) - \gamma_1 + 1, M(4)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(4)$. Also here it turns out that only very few transfers are possible.

1.) If $(s_2, s_3, 0) \in D^*$, $s_3 \geq 1$ is used we have

$$\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + s_3\gamma_2 \leq h + \delta.$$ 

Averaging with (59) gives $\epsilon_4 + 3\gamma_3/2 - 3\beta_2^{(4)}/2 \leq h + \delta$, contradicting (45) for $t = 3$ since $\beta_2^{(4)} \leq 2\gamma_2/3$.

2.) If $(s_2, 2, 1) \in E$ is used we have

$$\epsilon_4 + 2\gamma_3 + 4\beta_2^{(4)} \leq h + \delta.$$ 

Averaging with (59) gives $\epsilon_4 + 9\gamma_3/2 + 2\gamma_2 - 3\beta_2^{(4)}/2 \leq h + \delta$. This yields $\epsilon_4 - 2\gamma_2/2 + 2\gamma_2 \leq h + \delta$. contradicting (45) for $t = 3$.

3.) The minimal representation of $M(4)$ cannot be the regular one. Otherwise we would have

$$\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \gamma_1 \leq h + \delta.$$ 

Averaging with (59) gives $\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1/2 \leq h + \delta$. This yields $\epsilon_4 + 4\gamma_3 + \gamma_2/3 + \gamma_1/2 \leq h + \delta$ with a coefficient bound $1.39 < 2.008$ according to (41).

4.) If $(s_2, s_3, s_4) \in B^*$ is used we have $s_3 = 1 + \lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor$ and thus

$$\epsilon_4 + (s_4)\gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 - s_4\beta_2^{(4)} + (1 + \lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor)\gamma_2 \leq h + \delta.$$ 

Again averaging with (59) gives

$$\epsilon_4 + (4 + \frac{s_4}{2})\gamma_3 + \gamma_2 - \beta_2^{(4)} + \frac{(\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor + 1)\gamma_2 - s_4\beta_2^{(4)}}{2} \leq h + \delta.$$ 

This contradicts (45) for $t = 5$ if $s_4 \geq 4$ since $(\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor + 1)\gamma_2 - s_4\beta_2^{(4)} \geq 0$ and $\gamma_2 - \beta_2^{(4)} \geq \gamma_2/3$. For $s_4 = 2$ we have $\epsilon_4 + 5\gamma_3 + 2\gamma_2/3 \leq h + \delta$ contradicting (45) for $t = 4$. Thus only $s_4 = 3$ and $s_4 = 1$ may be possible.

5.) We now check whether If $(s_2, s_3, s_4) \in A$ can be used. we have $s_3 = \lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor$. Of course $s_4 = 4$ cannot be used. Otherwise we get a negative second coefficient. Since

$$4\beta_2^{(4)} - 2\gamma_2 < \beta_2^{(4)}$$

$$4\beta_2^{(4)} - 2\gamma_2 < 3\beta_2^{(4)} - \gamma_2$$

$$4\beta_2^{(4)} - 2\gamma_2 < 6\beta_2^{(4)} - 3\gamma_2$$
neither $s_4 = 1, s_4 = 3$ nor $s_4 = 6$ can be used. For $s_4 = 5$ we get

$$\epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} \leq h + \delta.$$ 

contradicting (45) for $t = 5$ since $\beta_2^{(4)} \leq 2\gamma_2/3$ and we are left with the possibility $s_4 = 2$.

**Table 7. The interval $I_8$**

| $K(4)$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $q$ | $d_1$ | $d_2$ | $d_3$ |
|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 2      | M    | 2    | M    | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2    |      |      |      |

All together the transfers $([1, 2], [1, 2]) \in A$, $([1, 1], [1, 1], [2, 3], [2, 3]) \in B^*$ and $(0, 0, 0)$ are possible. This gives us 8 cases to consider.

A) $(0, 0, 0)$. This cases is already treated under number 3 above.

B) $([1, 2], [1, 2])$ and $(0, 0, 0)$. This means

$$\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta$$

$$\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \kappa_1 \leq h + \delta.$$ 

Averaging again with (59) yields

$$\epsilon_4 + 11\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta.$$ 

yielding a coefficient $1.94 < 2.008$ according to (41) and (43).

C) $([1, 1], [1, 1])$ and $(0, 0, 0)$. This means

$$\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta$$

$$\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \kappa_1 \leq h + \delta.$$ 

Averaging again with (59) yields

$$\epsilon_4 + 10\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.$$ 

Since $\beta_2^{(4)} \geq 3\gamma_2/5$ we get $\epsilon_4 + 10\gamma_3/3 + 8\gamma_2/15 + \gamma_1/3 \leq h + \delta$ yielding a coefficient $1.59 < 2.008$ according to (41) and (43).

D) $([2, 3], [2, 3])$ and $(0, 0, 0)$. This means

$$\epsilon_4 + 4\gamma_3 + \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta$$

$$\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \kappa_1 \leq h + \delta.$$
Averaging again with (59) yields
\[ \epsilon_4 + 4\gamma_3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta. \]

Since \( \beta_2^{(4)} \leq 2\gamma_2/3 \) we get \( \epsilon_4 + 4\gamma_3 + 4\gamma_2/9 + \gamma_1/3 \leq h + \delta \) yielding a coefficient \( 1.51 < 2.008 \) according to (41) and (43).

E) \(([1, 1, 1],[2, 3], 2, 3)\) and \((0, 0, 0)\). This means
\[ \epsilon_4 + 4\gamma_3 + \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_2 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \kappa_2 \leq h + \delta. \]

Averaging again with (59) with equal weights this yields
\[ \epsilon_4 + 14\gamma_3/4 + (\gamma_2 + 2\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta. \]

Since \( \beta_2^{(4)} \geq 3\gamma_2/5 \) we get \( \epsilon_4 + 14\gamma_3/4 + 11\gamma_2/20 + \gamma_1/4 \leq h + \delta \) yielding a coefficient \( 1.75 < 2.008 \) according to (41) and (43). The other ordering of the transfers gives us the same average inequality and therefore need not to be considered separately.

F) \(([1, 1, 1],[1, 2],[2, 3])\) and \((0, 0, 0)\). This means
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_2 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \kappa_2 \leq h + \delta. \]

Using the weight distribution 1,1,1 and 2 for (59) we get
\[ \epsilon_4 + 4\gamma_3 + (4\gamma_2 - 3\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta. \]

Since \( \beta_2^{(4)} \leq 2\gamma_2/3 \) we get \( \epsilon_4 + 4\gamma_3 + 2\gamma_2/5 + \gamma_1/5 \leq h + \delta \) yielding a coefficient \( 2 < 2.008 \) according to (41) and (43). The other ordering of the transfers gives us the same average inequality and therefore need not to be considered separately.

G) \(([1, 2],[1, 2],[2, 3])\) and \((0, 0, 0)\). This means
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \beta_2^{(4)} + \kappa_1 - \kappa_2 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \kappa_2 \leq h + \delta. \]
Averaging again with (59) with equal weights this yields
\[ \epsilon_4 + 15\gamma_3/4 + (\gamma_2 + \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta. \]

Since \( \beta_2^{(4)} \geq 3\gamma_2/5 \) we get \( \epsilon_4 + 15\gamma_3/4 + 2\gamma_2/5 + \gamma_1/4 \leq h + \delta \) yielding a coefficient 1.97 < 2.008 according to (41) and (43). The other ordering of the transfers gives us the same average inequality and therefore need not to be considered separately.

H)((1, 2], 1, 2), ([1, 1], 1, 1), ([2, 3], 2, 3) and (0, 0, 0). This means
\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \beta_2^{(4)} + \gamma_1 - \kappa_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \kappa_1 - \kappa_2 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - \gamma_2 + \kappa_2 - \kappa_3 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \kappa_3 & \leq h + \delta.
\end{align*}
\]

Using the weight distribution 1,1,1,1 and 2 for (59) we get
\[ \epsilon_4 + 4\gamma_3 + (4\gamma_2 - 2\beta_2^{(4)})/6 + \gamma_1/6 \leq h + \delta. \]

Since \( \beta_2^{(4)} \leq 2\gamma_2/3 \) we get \( \epsilon_4 + 4\gamma_3 + 4\gamma_2/9 + \gamma_1/5 \leq h + \delta \) yielding a coefficient 1.9 < 2,008 according to (41) and (43). The other orderings of the transfers give us the same average inequality and therefore need not to be considered separately.

Thus we finished interval \( I_8 \).

### 11.3 Interval \( I_{11} \)

Now we have to study \( I_{11} \) where we have \( \lfloor \frac{6\beta_2^{(4)}}{2\gamma_2} \rfloor = 4 \) in (59) and our considerations will be similar to those from \( I_8 \). But we have to consider two new key numbers. We look at
\[
M(2) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (2\beta_2^{(4)} - \gamma_2 - 1)a_2 + (\gamma_1 - 1)
\]
and
\[
M(3) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (3\beta_2^{(4)} - 2\gamma_2 - 1)a_2 + (\gamma_1 - 1).
\]

We try to find representations for \( M(2) \) and all the numbers in the inverall \( K(2) = [M(2) - \gamma_1 + 1, M(2)] \) in addition to \( M(3) \) and all the numbers in the inverall \( K(3) = [M(3) - \gamma_1 + 1, M(3)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the intervals \( K(2) \) and \( K(3) \). Also here it turns out that only very few transfers are possible.
If a transfer $S$ cannot be used in the interval $K(3) = [M(3) - \gamma_1 + 1, M(3)]$ because of a coefficient bound below 2.008 then this transfer cannot be used in the interval $K(2) = [M(2) - \gamma_1 + 1, M(2)]$ either, since it would cause an even lower coefficient bound there. The coefficient $2\beta_2^{(4)} - \gamma_2 - 1$ is always larger than the coefficient $3\beta_2^{(4)} - 2\gamma_2 - 1$ for the corresponding number in $K(2)$. Therefore we are able to treat these two intervals, $K(3)$ and $K(2)$ simultaneously.

1.) If $(s_2, s_3, 0) \in D^*$, $s_3 \geq 1$ is used in $K(3)$ we have

$$\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + s_3\gamma_2 \leq h + \delta.$$ 

Averaging with (59) using the weight distribution 1 and 2 gives

$$\epsilon_4 + 5\gamma_3 + 3(\gamma_2 - \beta_2^{(4)}) \leq h + \delta,$$ 

contradicting (45) for $t = 4$ since $\beta_2^{(4)} \leq 5\gamma_2/6$. Thus the use of $(s_2, s_3, 0) \in D^*$, $s_3 \geq 1$ for numbers from $K(2)$ is also impossible.

2.) If $([2, 1], 2, 1) \in E$ is used in $K(3)$ we have

$$\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} \leq h + \delta.$$ 

Averaging with (59) gives

$$\epsilon_4 + 9\gamma_3/2 + 5\gamma_2/2 - 2\beta_2^{(4)} \leq h + \delta.$$ 

This yields $\epsilon_4 + 9\gamma_3/2 + 5\gamma_2/6 \leq h + \delta$, contradicting (45) for $t = 3$. Thus the use of $([2, 1], 2, 1) \in E$ for numbers from $K(2)$ is also impossible.

3.) If $(s_2, s_3, s_4) \in B^*$ is used for numbers in $K(3)$ we have $s_3 = 1 + \lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor$ and thus

$$\epsilon_4 + (1 + s_4)\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 - s_4\beta_2^{(4)} + (1 + \lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor)\gamma_2 \leq h + \delta.$$ 

Again averaging with (59) gives

$$\epsilon_4 + (4 + \frac{s_4}{2})\gamma_3 + 3(\gamma_2 - \beta_2^{(4)})/2 + \frac{(\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor + 1)\gamma_2 - s_4\beta_2^{(4)}}{2} \leq h + \delta.$$ 

This gives a coefficient bound $1.93 < 2.008$ according to (41) and (43) if $s_4 > 4$ since $(\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor + 1)\gamma_2 - s_4\beta_2^{(4)} \geq 0$ and $3(\gamma_2 - \beta_2^{(4)})/2 \geq \gamma_2/4$.

For $s_4 = 4$ we have $\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor = 3$ and thus

$$\epsilon_4 + 6\gamma_3 + 7\gamma_2/12 \leq h + \delta.$$ 

contradicting (45) for $t = 5$.

For $s_4 = 3$ we have $\lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor = 2$ and thus

$$\epsilon_4 + 11\gamma_3/2 + \gamma_2/2 \leq h + \delta.$$
contradicting (45) for \( t = 4 \). For \( s_4 = 2 \) and \( s_4 = 1 \) we cannot exclude the use of \((s_2, s_3, s_4) \in B^*\) when regarding numbers from \( K(3)\).

Nevertheless we manage to exclude the use of \((s_2, s_3, s_4) \in B^*\) for \( s_4 = 2 \) for interval \( K(2) \). There we have

\[
\epsilon_4 + 5\gamma_3 + 3(\gamma_2 - \beta_2^{(4)}) \leq h + \delta.
\]

contradicting (45) for \( t = 4 \) since \( 3(\gamma_2 - \beta_2^{(4)}) \geq \gamma_2 / 2 \).

4.) We now check whether \((s_2, s_3, s_4) \in A^*\) can be used for numbers from \( K(3) \).

We have \( s_3 = \lfloor \frac{s_4\beta_2^{(4)}}{\gamma_2} \rfloor \). Of course \( s_4 = 1, 2 \) and \( s_4 = 3 \) cannot be used.

Otherwise we get a negative second coefficient. For \( s_4 = 5 \) we get

\[
\epsilon_4 + 6\gamma_3 + 2(\gamma_2 - \beta_2^{(4)}) \leq h + \delta.
\]

contradicting (45) for \( t = 5 \) and for \( s_4 = 6 \) again we get a negative second coefficient since \( 3\beta_2^{(4)} - 2\gamma_2 < 6\beta_2^{(4)} - 4\gamma_2 \). Thus \( s_4 = 4 \) is the only possibility.

When considering numbers from \( K(2) \) we see that the transfers \((s_2, s_3, s_4) \in A\) are impossible for \( s_4 = 1, 2 \) because of the negative second coefficient. The cases \( s_4 = 3, 4 \) may be possible whilst \( s_4 = 5 \) gives

\[
\epsilon_4 + 6\gamma_3 + 3(\gamma_2 - \beta_2^{(4)}) \leq h + \delta.
\]

contradicting (45) for \( t = 5 \). For \( s_4 = 6 \) again we get a negative second coefficient since \( 2\beta_2^{(4)} - \gamma_2 < 6\beta_2^{(4)} - 4\gamma_2 \).

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<th>(r_3)</th>
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</tr>
<tr>
<td>(n_{k}A_{4(h)}) (\frac{1}{(h/4)^4}) (\leq)</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>[K(2)]</th>
<th>(a_2)- coef</th>
<th>-</th>
<th>-</th>
<th>-</th>
<th>-</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(n_{k}A_{4(h)}) (\frac{1}{(h/4)^4}) (\leq)</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

All together the transfers \(([1, 1], 1, 1), ([2, 2], 2, 2) \in B^*, ([3, 4], 3, 4) \in A^*\) and \((0, 0, 0)\) are possible for numbers from \( K(3) \) and \(([1, 1], 1, 1) \in B^*, ([2, 3], 2, 3), ([3, 4], 3, 4) \in A^*\) and \((0, 0, 0)\) are possible for numbers from \( K(2) \).

This again gives us a number of cases to consider. Since \(([1, 1], 1, 1), ([3, 4], 3, 4)\) and \((0, 0, 0)\) are in common for the two situations we treat them simultaneously. A)\((0, 0, 0)\) in interval \( K(3) \). This means

\[
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 \leq h + \delta.
\]
Averaging with (59) gives
\[ \epsilon_4 + 4\gamma_3 + 3(\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta. \]
This gives a coefficient bound 1.89 < 2.008 according to (41) and (43). Thus the use of the regular representation as the minimal representation for \( M(2) \) is also impossible.

B) \([1, 1], 1, 1\) and \(0, 0, 0\) in interval \(K(3)\). This means
\[ \epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - \kappa_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \kappa_1 \leq h + \delta. \]

Averaging with (59) with equal weights this yields
\[ \epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
yielding a coefficient 1.97 < 2.008 according to (41) since \( \beta_2^{(4)} \leq 5\gamma_2/6 \). Thus, trying to represent all numbers in \( K(2) \) only with \((1, 1, [1, 1])\) and \(0, 0, 0\) is also impossible.

C) \([3, 4], 3, 4\) and \(0, 0, 0\) in interval \(K(3)\). This means
\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \kappa_1 \leq h + \delta. \]

Averaging gives
\[ \epsilon_4 + 3\gamma_3 + (2\beta_2^{(4)} - \gamma_2)/2 + \gamma_1/2 \leq h + \delta. \]
yielding a coefficient 1.98 < 2.008 according to (41) and (43). Thus, trying to represent all numbers in \( K(2) \) only with \([3, 4], 3, 4\) and \(0, 0, 0\) is also impossible.

D) \((1, 1], 1, 1\), \([3, 4], 3, 4\) and \(0, 0, 0\). This case is possible for interval \(K(3)\). But we manage to show that it is impossible for \( K(2) \). There we have
\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \kappa_1 - \kappa_2 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \kappa_2 \leq h + \delta. \]

Averaging again gives
\[ \epsilon_4 + 8\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta. \]

Since \( \beta_2^{(4)} \geq 4\gamma_2/5 \) we get \( \epsilon_4 + 8\gamma_3/3 + 3\gamma_2/5 + \gamma_1/3 \leq h + \delta \) yielding a coefficient 1.81 < 2.008 according to (41) and (43).

The results so far are listet in the following table.
Table 9. The interval $I_{11}$

<table>
<thead>
<tr>
<th>Case</th>
<th>$K(3)$</th>
<th>$r_4$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>Case</th>
<th>$K(2)$</th>
<th>$r_4$</th>
<th>$q_1$</th>
<th>$r_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Impossible</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>A</td>
<td>Impossible</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>Impossible</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>C</td>
<td>Impossible</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>Impossible</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>B</td>
<td>Impossible</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>Possible</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>D</td>
<td>Impossible</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The remaining cases have to be treated separately for the intervals $K(3)$ and $K(2)$.

E) ($[2, 2], 2, 2$) and $(0, 0, 0)$ in $K(3)$. This means

$$
e_4 + 3\gamma_3 + \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta$$
$$
e_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \kappa_1 \leq h + \delta.$$

Averaging yields

$$
e_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1/2 \leq h + \delta.$$

Since $\beta_2^{(4)} \geq 4\gamma_2/5$ we get $e_4 + 2\gamma_3 + 3\gamma_2/5 + \gamma_1/2 \leq h + \delta$ yielding a coefficient $1.67 < 2.008$ according to (41).

F) ($[1, 1], 1, 1$), ($[2, 2], 2, 2$) and $(0, 0, 0)$ in $K(3)$. This means

$$
e_4 + 3\gamma_3 + \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta$$
$$
e_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \kappa_1 - \kappa_2 \leq h + \delta$$
$$
e_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \kappa_2 \leq h + \delta.$$

Averaging again with (59) with equal weights this yields

$$
e_4 + 13\gamma_3/4 + \gamma_2/2 + \gamma_1/4 \leq h + \delta.$$

yielding a coefficient $2 < 2.008$ according to (41) and (43). The other ordering gives the same inequality and the same coefficient.

G) ($[2, 2], 2, 2$), ($[3, 4], 3, 4$) and $(0, 0, 0)$ in $K(3)$. This means

$$
e_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta$$
$$
e_4 + 3\gamma_3 + \beta_2^{(4)} + \kappa_1 - \kappa_2 \leq h + \delta$$
$$
e_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \kappa_2 \leq h + \delta.$$

Averaging again yields

$$
e_4 + 3\gamma_3 + (3\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 \leq h + \delta.$$
Since \( \beta_2^{(4)} \geq 4\gamma_2/5 \) we get \( \epsilon_4 + 3\gamma_3 + 7\gamma_2/15 + \gamma_1/3 \leq h + \delta \) yielding a coefficient 1.95 < 2.008 according to (41) and (43). The other ordering of the transfers gives us the same average inequality and therefore need not to be considered separately.

H)(([1, 1], 1, 1), ([2, 2], 2, 2), ([3, 4], 3, 4) and (0, 0, 0) in \( K(3) \). This case cannot be excluded by the same methods, so we still have to consider it.

I) ([2, 3], 2, 3) and (0, 0, 0) in \( K(2) \). This means

\[
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \kappa_1 \leq h + \delta.
\]

Averaging yields

\[
\epsilon_4 + 5\gamma_3/2 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta.
\]

Since \( \beta_2^{(4)} \geq 4\gamma_2/5 \) we get \( \epsilon_4 + 5\gamma_3/2 + 2\gamma_2/5 + \gamma_1/2 \leq h + \delta \) yielding a coefficient 1.96 < 2.008 according to (41).

J)(([2, 3], 2, 3), ([1, 1], 1, 1) and (0, 0, 0) in \( K(2) \). This case cannot be excluded by the same methods, so we still have to consider it.

K)(([2, 3], 2, 3), ([3, 4], 3, 4) and (0, 0, 0) in \( K(2) \). This means

\[
\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \kappa_1 - \kappa_2 \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \kappa_2 \leq h + \delta.
\]

Averaging again yields

\[
\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
\]

Since \( \beta_2^{(4)} \leq 5\gamma_2/6 \) we get \( \epsilon_4 + 10\gamma_3/3 + 7\gamma_2/18 + \gamma_1/3 \leq h + \delta \) yielding a coefficient 1.97 < 2.008 according to (41) and (43). The other ordering of the transfers gives us the same average inequality and therefore need not to be considered.

Table 10. The interval \( I_{11} \) continuation

<table>
<thead>
<tr>
<th>Case</th>
<th>( K(3) )</th>
<th>( r_4 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>Case</th>
<th>( K(2) )</th>
<th>( r_4 )</th>
<th>( q_1 )</th>
<th>( r_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>Impossible</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>I</td>
<td>Impossible</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>Impossible</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>K</td>
<td>Impossible</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>F</td>
<td>Impossible</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>J</td>
<td>Possible</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>H</td>
<td>Possible</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>L</td>
<td>Possible</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
This gives us two possible sets of transfers for each of the intervals $K(2)$ and $K(3)$. We start with case L) where all four transfers $((1,1), (2,3), (3,4), (0,0))$ are used in $K(2)$.

This case again is possible. This means that the length of the corresponding list is 4. By the additional contribution argument (section 10.1) we may look at the average inequality for $K(3)$ and add the term $2(\gamma_2 - \beta_2^{(4)})$ on the left hand side. In the case of $K(3)$ with $(3,4), (1,1), (2,2), (0,0)$ we get

$$\epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta$$
$$\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \kappa_1 - \kappa_2 \leq h + \delta$$
$$\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + \kappa_2 - \kappa_3 \leq h + \delta$$
$$\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \kappa_3 \leq h + \delta.$$

The average inequality reads

$$\epsilon_4 + 11\gamma_3/4 + (5\beta_2^{(4)} - 2\gamma_2)/4 + \gamma_1/4 \leq h + \delta.$$

Other orderings of the transfers still leave us with the same average inequality. With the additional contribution this gives

$$\epsilon_4 + 11\gamma_3/4 + 3\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta.$$

Averaging with (59) using the weight distribution 8 and 1 gives

$$\epsilon_4 + 29\gamma_3/9 + 5\gamma_2/9 + 2\gamma_1/9 \leq h + \delta$$
with a coefficient bound $1.99 < 2.008$ according to (41) and (43). Thus the combination of H and L is impossible.

In the case D for $K(3)$ where $(3,4), (1,1), (2,2), (0,0)$ are used, we get

$$\epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \kappa_1 \leq h + \delta$$
$$\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \kappa_1 - \kappa_2 \leq h + \delta$$
$$\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \kappa_3 \leq h + \delta.$$

Averaging yields

$$\epsilon_4 + 8\gamma_3/3 + (4\beta_2^{(4)} - 2\gamma_2)/3 + \gamma_1/3 \leq h + \delta.$$

With the additional contribution this gives

$$\epsilon_4 + 8\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta.$$

yielding $\epsilon_4 + 8\gamma_3/3 + 8\gamma_2/15 + \gamma_1/3 \leq h + \delta$ with a coefficient bound $1.99 < 2.008$ according to (41) and (43). Thus the combination of D and L is also impossible and we have finished case L.
Thus we are left with the situation J where the transfers $([1,1], [1,1), ([2,3], 2,3)$ and $(0,0,0)$ are used in $K(2)$. Here the additional contribution argument is not strong enough to save us since the corresponding list is shorter and we need to go deeper into the cases and look at the inequality systems.

JD_a) In $K(2)$ we use the ordering $([2,3], [2,3), ([1,1], [1,1)$ and $(0,0,0)$

$$
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 & - \beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2,3]h_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2,3]h_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1,1]h_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1,1]h_1 & \leq h + \delta.
\end{align*}
$$

In $K(3)$ then we have to use the ordering $([3,4], [3,4), ([1,1], [1,1)$ and $(0,0,0)$, since the only place where new transfers can enter the list for $K(3)$ is the position of $([2,3], 2, 3)$ in the list for $K(2)$. Thus we have

$$
\begin{align*}
\epsilon_4 + 5\gamma_3 + \gamma_2 & - \beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 3\beta_1^{(3)} + [3,4]h_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 4\beta_1^{(4)} + 3\beta_1^{(3)} - [3,4]h_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1,1]h_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1,1]h_1 & \leq h + \delta.
\end{align*}
$$

Here we combine line 1 from the first inequality system with line 2 from the second one and get

$$
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)}/2 + (1 + [2,3] - [3,4] + [1,1])h_1/2 \leq h + \delta
$$

Now $(1 + [2,3] - [3,4] + [1,1])h_1/2$ is the average of two positive constant terms, so it has to be positive, i.e. $(1 + [2,3] - [3,4] + [1,1])h_1/2 \geq \gamma/2$ and we have

$$
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta
$$

yielding a coefficient bound $1.63 < 2.008$ according to (41) and (43) since $\beta_1^{(2)} \geq 4\gamma_2/5$.

JD_b) Now we reverse the ordering in $K(2)$. Thus we have $([1,1], [1,1), ([2,3], 2,3) and (0,0,0)$ giving

$$
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1,1]h_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1,1]h_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2,3]h_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2,3]h_1 & \leq h + \delta.
\end{align*}
$$

In $K(3)$ then we have to use the ordering $([1,1], [1,1), ([3,4], 3,4) and (0,0,0)$, since the only place where new transfers can enter the list for $K(3)$ is the
position of ([2, 3], 2, 3) in the list for \( K(2) \). Thus we have
\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]_1 \gamma_1 - 4\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 4]_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 4\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 4]_1 & \leq h + \delta.
\end{align*}
\]
Here we combine the last line from the first inequality system with line 2 from the second one and get the same inequality as above and we are finished.

Thus we have finished case D and we now look at the longer list for \( K(3) \), ie. case H.

JHa) In \( K(2) \) we use the ordering ([2, 3], 2, 3), ([1, 1], 1, 1) and (0, 0, 0).
\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} & \leq [2, 3]_1 \gamma_1 \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]_1 \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} & \leq [1, 1]_1 \gamma_1 \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]_1 & \leq h + \delta.
\end{align*}
\]
In \( K(3) \) we now have two choices. We know that the transfer ([1, 1], 1, 1) has to occur in the second last line. But both orderings of the two new transfers have to be considered. First we use the ordering ([3, 4], 3, 4), ([2, 2], 2, 2), ([1, 1], 1, 1) and (0, 0, 0) giving
\[
\begin{align*}
\epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 3\beta_1^{(3)} & \leq [3, 4]_1 \gamma_1 \\
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 4]_1 \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} & \leq [2, 2]_1 \gamma_1 \\
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]_1 \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} & \leq [1, 1]_1 \gamma_1 \\
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]_1 & \leq h + \delta.
\end{align*}
\]
Here we combine line 1 and 3 from the first inequality system with line 2 from the second one and get
\[
\begin{align*}
\epsilon_4 + 8\gamma_3/3 + 2\beta_2^{(4)}/3 + (1 + [2, 3] - [1, 1] - [3, 4] + [2, 2])_1 \gamma_1/3 & \leq h + \delta
\end{align*}
\]
Again the constant term has to be positive and we get
\[
\begin{align*}
\epsilon_4 + 8\gamma_3/3 + 8\gamma_2/15 + \gamma_1/3 & \leq h + \delta
\end{align*}
\]
since \( \beta_2^{(4)} \geq 4\gamma_2/5 \). This yields a coefficient bound 1.98 < 2.008 according to (41) and (43).

JHb) The other ordering for \( K(3) \) namely ([2, 2], 2, 2), ([3, 4], 3, 4), ([1, 1], 1, 1) and (0, 0, 0) gives
\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} & \leq [2, 2]_1 \gamma_1 \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]_1 \gamma_1 - 4\beta_1^{(4)} - 3\beta_1^{(3)} & \leq [3, 4]_1 \gamma_1 \\
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 4\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 4]_1 \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} & \leq [1, 1]_1 \gamma_1 \\
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} & \leq [1, 1]_1 \gamma_1 \leq h + \delta.
\end{align*}
\]
Thus we finished

Above we saw that this leads to a coefficient bound below 1.

Here we combine line 2 from the first inequality system (for $K(2)$) with line 3 from the second one and get again

$$\epsilon_4 + 3\gamma_3 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta,$$

since the constant term has to be positive. Above we saw that this leads to a coefficient bound below 2.

**JHc)** In $K(2)$ we use the ordering $([1, 1], [1, 1], [2, 3], 2, 3)$ and $(0, 0, 0)$ giving

$$
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 & \leq h + \delta.
\end{align*}
$$

In $K(3)$ we have to start with $([1, 1], 1, 1)$. But then again we have two choices. First we use the ordering $([1, 1], [1, 1], [3, 4], 3, 4), ([2, 2], 2, 2)$ and $(0, 0, 0)$ giving

$$
\begin{align*}
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - \gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]\gamma_1 - 4\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 4]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]\gamma_1 & \leq h + \delta.
\end{align*}
$$

Here we combine line 2 from the first inequality system (for $K(2)$) with line 3 from the second one and get

$$\epsilon_4 + 7\gamma_3/2 + \gamma_2/2 + \gamma_1/3 \leq h + \delta$$

since the constant term has to be positive and we get a coefficient bound $1.14 < 2.008$ according to $(41)$ and $(43)$.

**JHd)** The other ordering for $K(3)$ namely $([1, 1], 1, 1), ([2, 2], 2, 2), ([3, 4], 3, 4)$ and $(0, 0, 0)$ gives

$$
\begin{align*}
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]\gamma_1 - 4\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 4\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 4]\gamma_1 & \leq h + \delta.
\end{align*}
$$

Here we combine line 2 and 3 from the first inequality system (for $K(2)$) with line 2 from the second one and get again

$$\epsilon_4 + 8\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta.$$

Above we saw that this leads to a coefficient bound below $1.98 < 2.008$.

Thus we finished $I_{11}$. 
11.4 $r_6 = r_5 = r_4 = 0$ in Interval $I_{12}$

Our next goal is to show $r_6 = r_5 = r_4 = 0$ in the main list of $I_{12}$. This is a formidable task and we have to go into the details of the different orderings of the transfers.

If we assume that $([5, 6], 5, 6)$ has been used in the main list this gives the inequality
\[ \epsilon_4 + 7\gamma_3 + 6\gamma_2 - 6\beta_2^{(4)} \leq h + \delta, \]  
where we left out the constant term.

If we assume that $([4, 5], 4, 5)$ has been used in the main list this gives the inequality
\[ \epsilon_4 + 6\gamma_3 + 5\gamma_2 - 5\beta_2^{(4)} \leq h + \delta, \]  
where we left out the constant term.

If we assume that $([3, 4], 3, 4) \in A$ has been used in the main list this gives the inequality
\[ \epsilon_4 + 5\gamma_3 + 4\gamma_2 - 4\beta_2^{(4)} \leq h + \delta. \]  
Here (62) is the weakest of these three inequalities. If we manage to show that (62) is impossible, we have shown that all three inequalities are impossible.

Here we study the key number

\[ M(4) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (4\beta_2^{(4)} - 3\gamma_2 - 1)a_2 + (\gamma_1 - 1) \]

and try to find a representation for $M(4)$ and all the numbers in the interval $K(4) = [M(4) - \gamma_1 + 1, M(4)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(4)$. Also here it turns out that only very few transfers are possible.

1.) If $(s_2, s_3, 0) \in D^*, s_3 \geq 2$ is used we have
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + s_3\gamma_2 \leq h + \delta. \]

Averaging with (62) gives $\epsilon_4 + 3\gamma_3 + 3\gamma_2/2 \leq h + \delta$, contradicting (45) for $t = 2$. But the case $s_3 = 1$ has still to be considered.

2.) If $(s_2, 2, 1) \in E$ is used we have
\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - \gamma_2 \leq h + \delta. \]

Averaging with (62) gives $\epsilon_4 + 7\gamma_3/2 + 3\gamma_2/2 - \beta_2^{(4)}/2 \leq h + \delta$. This yields $\epsilon_4 + 7\gamma_3/2 + \gamma_2 \leq h + \delta$, yielding a coefficient $1.92 < 2.008$ according to (41) and (43).

3.) The minimal representation of $M(4)$ cannot be the regular one. Otherwise we would have
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \gamma_1 \leq h + \delta. \]
Averaging with (62) gives $\epsilon_4 + 3\gamma_3 + \gamma_2/2 + \gamma_1/2 \leq h + \delta$ with a coefficient bound $1.33 < 2.008$ according to (41).

4.) If $(s_2, s_3, s_4) \in B^*$ is used we have $s_3 = 1 + \lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \rfloor$ and thus

$$\epsilon_4 + (1 + s_4)\gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 - s_4\beta_2^{(4)} + (1 + \lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \rfloor)\gamma_2 \leq h + \delta.$$  Again averaging with (62) gives

$$\epsilon_4 + (3 + \frac{s_4}{2})\gamma_3 + \gamma_2/2 + \frac{((\frac{s_4 \beta_2^{(4)}}{\gamma_2} + 1)\gamma_2 - i\beta_2^{(4)})}{2} \leq h + \delta.$$

This contradicts (45) for $t = 4$ if $s_4 \geq 4$ since $((\frac{s_4 \beta_2^{(4)}}{\gamma_2} + 1)\gamma_2 - s_4\beta_2^{(4)}) \geq 0$. For $s_4 = 3$ we have $\epsilon_4 + 4\gamma_3 + \beta_2^{(4)} \leq h + \delta$ giving $\epsilon_4 + 4\gamma_3 + 5\gamma_2/6 \leq h + \delta$, yielding a coefficient $1.89 < 2.008$ according to (41) and (43). Thus we are left with the possibilities $s_4 = 1$ and $s_4 = 2$.

5.) We now check whether $(s_2, s_3, s_4) \in A$ can be used. We have $s_3 = \lfloor \frac{s_4 \beta_2^{(4)}}{\gamma_2} \rfloor$. Of course $s_4 \leq 4$ cannot be used. Otherwise we get a negative second coefficient. Thus we are left with $s_4 = 5$ and $s_4 = 6$.

Here the transfers $(0, 1, 0) \in D^*$, $([1, 1], 1, 1), ([2, 2], 2, 2) \in B^*$ and $([4, 5], 4, 5), ([5, 6], 5, 6) \in A$ could not be excluded and therefore 32 cases arise:
The transfers $r_6 = 1$ and $q_2 = 1$ cannot appear at the same time. Otherwise we would have

$$
\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} \leq h + \delta
$$

$$
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 \leq h + \delta
$$

(63)

\[ 2.054 \]

\[ 2.009 \]

\[ 2.040 \]

\[ 1.98 \]

\[ 1.98 \]

\[ 1.98 \]

\[ 1.98 \]

\[ 1.98 \]

\[ 1.98 \]

\[ 1.97 \]

\[ 1.97 \]

Table 11. $r_4 = 1$ in interval $I_{12}$

<table>
<thead>
<tr>
<th>$r_5$</th>
<th>$r_6$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$d_1$</th>
<th>Average inequality with (62)</th>
<th>Weights</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3 + 2\gamma_2 + \gamma_1/2 \leq h + \delta$</td>
<td>1, 1</td>
<td>1.33</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 3\gamma_3 + 2\gamma_2 + \gamma_1/2 \leq h + \delta$</td>
<td>1.1, 2</td>
<td>1.61</td>
</tr>
<tr>
<td>0</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 23\gamma_3/7 + 4\gamma_2/7 + 2\gamma_1/7 \leq h + \delta$</td>
<td>2.2, 3</td>
<td>1.68</td>
</tr>
<tr>
<td>0</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 37\gamma_3/11 + 8\gamma_2/11 + 2\gamma_1/11 \leq h + \delta$</td>
<td>2.2, 2, 5</td>
<td>1.33</td>
</tr>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 47\gamma_3/15 + 8\gamma_2/15 + 4\gamma_1/15 \leq h + \delta$</td>
<td>14.4, 7</td>
<td>1.92</td>
</tr>
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<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 71\gamma_3/23 + 16\gamma_2/23 + 4\gamma_1/23 \leq h + \delta$</td>
<td>4.4, 4, 11</td>
<td>1.96</td>
</tr>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 69\gamma_3/21 + 12\gamma_2/21 + 4\gamma_1/21 \leq h + \delta$</td>
<td>4.4, 4, 9</td>
<td>2.029</td>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 21\gamma_3/5 + 2\gamma_2/5 + 2\gamma_1/5 \leq h + \delta$</td>
<td>4.4, 4, 13</td>
<td>2.040</td>
</tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + 2\gamma_2/3 + 2\gamma_1/9 \leq h + \delta$</td>
<td>2.2, 2, 3</td>
<td>1.53</td>
</tr>
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<td>0</td>
<td>0</td>
<td>see (63) below</td>
<td>4.4, 4, 5</td>
<td>1.80</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td>see (63) below</td>
<td>4.4, 4, 4, 9</td>
<td>1.84</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>see (63) below</td>
<td>4.4, 4, 4, 9</td>
<td>1.84</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 43\gamma_3/11 + 4\gamma_2/11 + 4\gamma_1/11 \leq h + \delta$</td>
<td>4.4, 3</td>
<td>1.65</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 67\gamma_3/19 + 12\gamma_2/19 + 4\gamma_1/19 \leq h + \delta$</td>
<td>4.4, 4, 7</td>
<td>1.70</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 65\gamma_3/17 + 8\gamma_2/17 + 4\gamma_1/17 \leq h + \delta$</td>
<td>4.4, 4, 5</td>
<td>1.80</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 89\gamma_3/25 + 16\gamma_2/25 + 4\gamma_1/25 \leq h + \delta$</td>
<td>2.2, 2, 5</td>
<td>1.84</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3 + 4\gamma_2/11 + 2\gamma_1/11 \leq h + \delta$</td>
<td>2.2, 2, 3</td>
<td>2.15</td>
</tr>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 45\gamma_3/13 + 8\gamma_2/13 + 2\gamma_1/13 \leq h + \delta$</td>
<td>2.2, 2, 2, 5</td>
<td>1.97</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 11\gamma_3/3 + 2\gamma_2/2 + \gamma_1/6 \leq h + \delta$</td>
<td>1, 1, 1, 1, 2</td>
<td>2.054</td>
</tr>
<tr>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 28\gamma_3/8 + 5\gamma_2/8 + \gamma_1/8 \leq h + \delta$</td>
<td>1, 1, 1, 1, 1, 3</td>
<td>2.039</td>
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<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 61\gamma_3/13 + 4\gamma_2/13 + 4\gamma_1/13 \leq h + \delta$</td>
<td>4.4, 4, 1</td>
<td>1.63</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 85\gamma_3/21 + 12\gamma_2/21 + 4\gamma_1/21 \leq h + \delta$</td>
<td>4.4, 4, 4, 5</td>
<td>1.64</td>
</tr>
<tr>
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<td>0</td>
<td>see (63) below</td>
<td>4.4, 4, 4, 5</td>
<td>1.64</td>
</tr>
<tr>
<td>1</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>see (63) below</td>
<td>4.4, 4, 4, 5</td>
<td>1.64</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 21\gamma_3/5 + 2\gamma_2/5 + \gamma_1/5 \leq h + \delta$</td>
<td>1, 1, 1, 1, 1</td>
<td>1.90</td>
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<td>1</td>
<td>$\epsilon_4 + 27\gamma_3/7 + 4\gamma_2/7 + \gamma_1/7 \leq h + \delta$</td>
<td>1, 1, 1, 1, 1, 2</td>
<td>1.88</td>
</tr>
<tr>
<td>1</td>
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<td>0</td>
<td>see (63) below</td>
<td>1, 1, 1, 1, 1, 2</td>
<td>1.88</td>
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<td>1</td>
<td>1</td>
<td>see (63) below</td>
<td>1, 1, 1, 1, 1, 2</td>
<td>1.88</td>
</tr>
</tbody>
</table>

\[ 1.98 \]

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\[ 1.98 \]

\[ 1.98 \]
Four situations could not be settled in the table. Here we have to go through a number of different orderings in order to settle the case. These are the following:

A) \( ([1, 1], 1, 1), ([2, 2], 2, 2) \in B^* \) together with \( (0, 0, 0) \). Here we have two different orderings.

\( \alpha \) For the first ordering the inequality system looks like this:

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1] \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [1, 1] \gamma_1 & \leq h + \delta.
\end{align*}
\]

Averaging with (62) using the weights 0,2,1 and 2 gives

\[ \epsilon_4 + 17\gamma_3/5 + 3\gamma_2/5 + \gamma_1/5 \leq h + \delta. \]

yielding a coefficient 1.87 < 2.008 according to (41) and (43).

\( \beta \) The other ordering gives

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2] \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1] \gamma_1 & \leq h + \delta.
\end{align*}
\]

Again averaging with (62) using the weights 1,2,0 and 2 gives

\[ \epsilon_4 + 17\gamma_3/5 + 3\gamma_2/5 + \gamma_1/5 \leq h + \delta. \]

and we get the same coefficient 1.87 < 2.008 as above.

B) \( ([1, 1], 1, 1), ([2, 2], 2, 2) \in B^* \) together with \( (0, 1, 0) \in D^* \) and \( (0, 0, 0) \). If the ordering is \( (0, 1, 0,),(1,1,1,1),(2,2,2,2) \) we can use the same argument as in \( A\alpha \). If the ordering is \( (2, 2, 2, 2), ([1, 1], 2, 2), (0, 1, 0) \) we can use the same argument as in \( A\beta \). This means there are four cases left.

\( B\alpha \) \( ([1, 1], 1, 2), ([2, 2], 2, 2), (0, 1, 0) \) and \( (0, 0, 0) \)

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1] \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2] \gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]

Averaging with (62) using the weights 0, 2, 1, 1 and 3 gives

\[ \epsilon_4 + 23\gamma_3/7 + 5\gamma_2/7 + \gamma_1/7 \leq h + \delta. \]

yielding a coefficient 1.92 < 2.008 according to (41) and (43).
Using the weights $0, 2, 2, 1$ and $4$ gives

\[
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(3)} - [1, 1] \gamma_1 \leq h + \delta
\]

yielding a coefficient $1.92 < 2.008$ according to (41) and (43).

Averaging with (62) using the weights $0, 2, 2, 1, 4$ and $3$ gives

\[
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(3)} - [1, 1] \gamma_1 \leq h + \delta
\]

yielding a coefficient $1.97 < 2.008$ according to (41) and (43).

Averaging again with (62) using the weights $0, 2, 2, 1$ and $4$ gives

\[
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(3)} - [1, 1] \gamma_1 \leq h + \delta
\]

yielding a coefficient $1.97 < 2.008$ according to (41) and (43).

If the ordering is $(4, 5), (1, 1, 1, 1, 0, 0)$, we can use the same argument as in $A\beta$. This means there are four cases left.
Averaging again with (62) using the weights 4, 4, 8, 0 and 7 gives

\[ C_\alpha \ ([(1,1),1,1),([2,2],2,2),([4,5],4,5) \text{ and } (0,0,0) \]

\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1,1]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1,1]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2,2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 6\gamma_3 + \gamma_2 + \beta_2^{(4)} + 2\beta_1^{(3)} - [2,2]\gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4,5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4,5]\gamma_1 \leq h + \delta. \]

Averaging with (62) using the weights 0, 8, 4, 4 and 7 gives

\[ \epsilon_4 + 87\gamma_3/23 + 12\gamma_2/23 + 4\gamma_1/23 \leq h + \delta, \]

yielding a coefficient 1.92 < 2.008 according to (41) and (43).

\[ C_\beta \ ([(4,5],4,5),([2,2],2,2),([1,1],1,1) \text{ and } (0,0,0) \]

\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4,5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4,5]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2,2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2,2]\gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1,1]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1,1]\gamma_1 \leq h + \delta. \]

Averaging again with (62) using the weights 4, 4, 8, 0 and 7 gives

\[ \epsilon_4 + 87\gamma_3/23 + 12\gamma_2/23 + 4\gamma_1/23 \leq h + \delta, \]

yielding a coefficient 1.92 < 2.008 according to (41) and (43).

\[ C_\gamma \ ([(1,1],1,1),([4,5],4,5),([2,2],2,2) \text{ and } (0,0,0) \]

\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1,1]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1,1]\gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4,5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4,5]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2,2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2,2]\gamma_1 \leq h + \delta. \]

Averaging with (62) using the weights 0, 4, 4, 2 and 3 gives

\[ \epsilon_4 + 53\gamma_3/13 + 6\gamma_2/13 + 2\gamma_1/13 \leq h + \delta, \]

yielding a coefficient 1.97 < 2.008 according to (41) and (43).

\[ C_\delta \ ([(2,2],2,2),([4,5],4,5),([1,1],1,1) \text{ and } (0,0,0) \]

\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2,2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2,2]\gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4,5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - \gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4,5]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2,2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1,1]\gamma_1 \leq h + \delta. \]
Averaging again with (62) using the weights 2, 4, 4, 0 and 3 gives
\[ \epsilon_4 + 53\gamma_3/13 + 6\gamma_2/13 + 2\gamma_1/13 \leq h + \delta, \]
yielding a coefficient 1.97 < 2.008 according to (41) and (43). D) \([1, 1], [1, 1], [2, 2], [2, 2] \in B^* \) together with \([4, 5], 4, 5 \in A, (0, 1, 0) \in D^* \)
and \((0, 0, 0) \). In the table below you can see what arguments can be used to exclude the actual cases:

**Table 12. Arguments used to exclude the different orderings**

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_5 )</td>
<td>( q_1 )</td>
<td>( q_2 )</td>
<td>( d_1 )</td>
<td>( B\alpha )</td>
</tr>
<tr>
<td>( r_5 )</td>
<td>( q_1 )</td>
<td>( d_1 )</td>
<td>( q_2 )</td>
<td>( B\gamma )</td>
</tr>
<tr>
<td>( r_5 )</td>
<td>( d_1 )</td>
<td>( q_1 )</td>
<td>( q_2 )</td>
<td>( A\alpha )</td>
</tr>
<tr>
<td>( r_5 )</td>
<td>( d_1 )</td>
<td>( q_2 )</td>
<td>( q_1 )</td>
<td>( D\alpha ) see below</td>
</tr>
<tr>
<td>( r_5 )</td>
<td>( q_2 )</td>
<td>( d_1 )</td>
<td>( q_1 )</td>
<td>( D\beta ) see below</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>( r_5 )</td>
<td>( q_2 )</td>
<td>( q_1 )</td>
<td>( A\alpha )</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>( q_1 )</td>
<td>( q_2 )</td>
<td>( r_5 )</td>
<td>( C\alpha )</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>( q_1 )</td>
<td>( r_5 )</td>
<td>( q_2 )</td>
<td>( C\gamma )</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>( q_2 )</td>
<td>( r_5 )</td>
<td>( q_1 )</td>
<td>( B\beta )</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>( r_5 )</td>
<td>( q_1 )</td>
<td>( d_1 )</td>
<td>( D\gamma ) see below</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>( q_2 )</td>
<td>( r_5 )</td>
<td>( q_1 )</td>
<td>( D\delta ) see below</td>
</tr>
<tr>
<td>( r_5 )</td>
<td>( q_1 )</td>
<td>( q_2 )</td>
<td>( r_5 )</td>
<td>( D\epsilon ) see below</td>
</tr>
</tbody>
</table>

This means we are left with 12 cases.

**D**\( \alpha \) \([4, 5], 4, 5 \), \((0, 1, 0) \), \([2, 2], 2, 2 \), \([1, 1], 1, 1 \) and \((0, 0, 0) \)
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5] \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2] \gamma_1 \leq h + \delta \]

Averaging with (62) using the weights 4, 4, 4, 8, 0 and 11 gives
\[ \epsilon_4 + 20\gamma_2/31 + 4\gamma_1/31 \leq h + \delta, \]
yielding a coefficient 1.94 < 2.008 according to (41) and (43). D\( \beta \) \([4, 5], 4, 5 \), \([2, 2], 2, 2 \), \((0, 1, 0) \), \([1, 1, 1, 1) \) and \((0, 0, 0) \)
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5] \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]_1 \gamma_1 - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \beta_1^{(3)} - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]_1 \gamma_1 \leq h + \delta. \]

Averaging with \((62)\) using the weights 2, 4, 0, 2, 0 and 3 gives
\[ \epsilon_4 + 43\gamma_3/11 + 6\gamma_2/11 + 2\gamma_1/11 \leq h + \delta, \]
yielding a coefficient 1.78 < 2.008 according to \((41)\) and \((43)\).
\[ D\gamma \) \((0, 1, 0), ([4, 5], 4, 5), ([2, 2], 2, 2), ([1, 1], 1, 1) \) and \((0, 0, 0) \)
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(3)} - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]_1 \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]_1 \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]_1 \gamma_1 \leq h + \delta. \]

Averaging with \((62)\) using the weights 4, 4, 4, 8, 0 and 11 gives
\[ \epsilon_4 + 111\gamma_3/31 + 20\gamma_2/31 + 4\gamma_1/31 \leq h + \delta, \]
yielding a coefficient 1.94 < 2.008 according to \((41)\) and \((43)\).
\[ D\delta \) \((0, 1, 0), ([2, 2], 2, 2), ([4, 5], 4, 5), ([1, 1], 1, 1) \) and \((0, 0, 0) \)
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(3)} - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]_1 \gamma_1 - 5\beta_1^{(4)} + 4\beta_1^{(3)} + [4, 5]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]_1 \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]_1 \gamma_1 \leq h + \delta. \]

Averaging with \((62)\) using the weights 2, 2, 4, 4, 0 and 5 gives
\[ \epsilon_4 + 65\gamma_3/17 + 10\gamma_2/17 + 2\gamma_1/17 \leq h + \delta, \]
yielding a coefficient 1.96 < 2.008 according to \((41)\) and \((43)\).
\[ D\epsilon \) \(([1, 1], 1, 1), ([2, 2], 2, 2), (0, 1, 0), ([4, 5], 4, 5) \) and \((0, 0, 0) \)
\[ \epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(3)} + [1, 1]_1 \gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]_1 \gamma_1 - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(3)} - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5]_1 \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]_1 \gamma_1 \leq h + \delta. \]
Averaging with (62) using the weights 0, 8, 4, 4, 4 and 11 gives
\[ \epsilon_4 + 111\gamma_3/31 + 20\gamma_2/31 + 4\gamma_1/31 \leq h + \delta, \]
yielding a coefficient 1.95 < 2.008 according to (41) and (43).

Averaging with (62) using the weights 0, 2, 0, 4, 4, 4 and 11 gives again
\[ \epsilon_4 + 2\gamma_3 + 3\beta_1^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 \leq h + \delta, \]
\[ \epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]\gamma_1 \leq h + \delta, \]
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]\gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5]\gamma_1 \leq h + \delta, \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]\gamma_1 - \beta_1^{(3)} \leq h + \delta, \]
\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(3)} \leq h + \delta. \]

Averaging with (62) using the weights 0, 8, 4, 4, 4 and 11 gives again
\[ \epsilon_4 + 111\gamma_3/31 + 20\gamma_2/31 + 4\gamma_1/31 \leq h + \delta, \]
yielding a coefficient 1.95 < 2.008 according to (41) and (43).

Averaging with (62) using the weights 0, 2, 0, 4, 2, and 3 gives again
\[ \epsilon_4 + 43\gamma_3/11 + 6\gamma_2/11 + 2\gamma_1/11 \leq h + \delta, \]
yielding a coefficient 1.78 < 2.008 according to (41) and (43).

Averaging with (62) using the weights 2, 0, 2, 0, 4, 0, and 3 gives again
\[ \epsilon_4 + 43\gamma_3/11 + 6\gamma_2/11 + 2\gamma_1/11 \leq h + \delta, \]
yielding a coefficient $1.78 < 2.008$ according to (41) and (43).

$D\mu$ \(([1, 1], 1, 1), ([4, 5], 4, 5), (0, 1, 0), ([2, 2], 2, 2)\) and \((0, 0, 0)\)

\[
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]\gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]\gamma_1 \leq h + \delta
\]

Averaging with (62) using the weights 0, 4, 4, 2, and 7 gives again

\[
\epsilon_4 + 77\gamma_3/21 + 14\gamma_2/21 + 2\gamma_1/21 \leq h + \delta,
\]

yielding a coefficient $1.97 < 2.008$ according to (41) and (43).

$D\nu$ \(([1, 1], 1, 1), ([4, 5], 4, 5), ([2, 2], 2, 2), (0, 1, 0)\) and \((0, 0, 0)\)

\[
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 6\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]\gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - 2\gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]\gamma_1 - 2\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 2]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]\gamma_1 - \beta_1^{(3)} \leq h + \delta
\]

Averaging with (62) using the weights 0, 4, 4, 2, and 7 gives again

\[
\epsilon_4 + 77\gamma_3/21 + 14\gamma_2/21 + 2\gamma_1/21 \leq h + \delta,
\]

yielding a coefficient $1.97 < 2.008$ according to (41) and (43).

$D\xi$ \(([2, 2], 2, 2), (0, 1, 0), ([4, 5], 4, 5), ([1, 1], 1, 1)\) and \((0, 0, 0)\)

\[
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - 2\beta_2^{(4)} - 2\beta_1^{(3)} + [2, 2]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 3\gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]\gamma_1 - \beta_1^{(3)} \leq h + \delta
\]
\[
\epsilon_4 + 6\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]\gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + 2\beta_1^{(4)} + \beta_1^{(3)} - [2, 2]\gamma_1 \leq h + \delta
\]

Averaging with (62) using the weights 2, 4, 4, 0, and 7 gives again

\[
\epsilon_4 + 77\gamma_3/21 + 14\gamma_2/21 + 2\gamma_1/21 \leq h + \delta,
\]

yielding a coefficient $1.97 < 2.008$ according to (41) and (43).
\[ D_P ([2, 2], 2, 2)([4, 5], 4, 5), ([1, 1], 1, 1), (0, 1, 0) \text{ and } (0, 0, 0) \]

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - 2\beta^{(4)} - 2\beta_1^{(3)} + [2, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 2]\gamma_1 - 5\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + 5\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 5]\gamma_1 - \beta_1^{(4)} - \beta_1^{(3)} + [1, 1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 2\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} - [1, 1]\gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - 3\gamma_2 + \beta_1^{(4)} + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]

Averaging with (62) using the weights 2, 4, 4, 0, 0, and 3 gives again

\[
\epsilon_4 + 53\gamma_3/13 + 6\gamma_2/13 + 2\gamma_1/13 \leq h + \delta,
\]

yielding a coefficient 1.96 < 2.008 according to (41) and (43).

Thus we have shown that the use of the transfers ([3, 4], 3, 4), ([4, 5], 4, 5) and ([5, 6], 5, 6) in the interval \( I_{12} \) are not possible. This fact reduces the amount of cases to consider remarkably.

### 11.5 Consequences for key numbers of higher order in Interval \( I_{12} \)

The computer run of the first programme showed that in all surviving main lists in \( I_{12} \) either the transfers \((0, 0, 1)\) or \(([1, 2], 1, 2)\) or both were used. We now want to exclude the use of \( r_6 \) in the list for \( M(1) \) ie. \( K(1) \) and \( M(2) \) ie. \( K(2) \). If \( r_6 = 1 \) was used for \( K(1) \) we would have

\[
\epsilon_4 + 7\gamma_3 + 5\gamma_2 - 5\beta_2^{(4)} \leq h + \delta, \tag{64}
\]

where we, as usual, left out the constant term.

If \( r_6 = 1 \) was used for \( K(2) \) we would have

\[
\epsilon_4 + 7\gamma_3 + 4\gamma_2 - 4\beta_2^{(4)} \leq h + \delta, \tag{65}
\]

where we, as usual, left out the constant term. Both inequalities imply (62) and thus they are impossible.

We may even exclude the use of \( r_5 \) in \( K(1) \) in interval \( I_{12} \) since this would imply

\[
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 4\beta_2^{(4)} \leq h + \delta, \tag{66}
\]

which would imply (62).
11.6 \( d_4 = 0 \) for key numbers of higher order in all intervals

In the section about the key numbers of higher order we saw that we had to extend \( D \) to \( D^* \) including \( d_4 \) ie. the transfer \((4,0), (4,0)\). We now show that this transfer cannot be used for any of the key numbers. If \( d_4 \) is used we have

\[
\epsilon_4 + \gamma_3 + 4\gamma_2 \leq h + \delta. \tag{67}
\]

Now if \( r_i \) with \( i \geq 2 \) or \( q_i \) with \( i \geq 2 \) is used in the main list we get

\[
\epsilon_4 + 3\gamma_3 \leq h + \delta.
\]

Averaging these two inequalities yields

\[
\epsilon_4 + 2\gamma_3 + 2\gamma_2 \leq h + \delta
\]

contradicting (45) for \( t = 2 \).

Inspecting all the main lists for all intervals we found only 7 cases that did not contain any \( r_i \) with \( i \geq 2 \) or \( q_i \) with \( i \geq 2 \). These cases are listed below:

1) \( r_1 \) and \((0,0,0)\).
2) \( r_1, d_1 \) and \((0,0,0)\).
3) \( r_1, q_1 \) and \((0,0,0)\).
4) \( r_1, d_1, d_2 \) and \((0,0,0)\).
5) \( r_1, q_1, d_1 \) and \((0,0,0)\).
6) \( r_1, q_1, d_2 \) and \((0,0,0)\).
7) \( r_1, q_1, d_1, d_2 \) and \((0,0,0)\).

In all these cases we combine the average inequality with (67) with equal weights. The corresponding coefficients will all be below 2.008. Here we do not need the interval bounds for \( \beta_2^{(4)} \). We only use \( \beta_2^{(4)} \leq \gamma_2 \).

1) \( r_1 \) and \((0,0,0)\). Here the average inequality reads

\[
\epsilon_4 + 3\gamma_3/2 + (2\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.
\]

Averaging with (67) gives

\[
\epsilon_4 + 5\gamma_3/4 + 9\gamma_2/4 + \gamma_1/4 \leq h + \delta
\]

yielding a coefficient 1.40 < 2.008 according to (41) and (43).

2) \( r_1, d_1 \) and \((0,0,0)\). Here the average inequality reads

\[
\epsilon_4 + 4\gamma_3/3 + (4\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
\]

Averaging with (67) gives

\[
\epsilon_4 + 7\gamma_3/6 + 5\gamma_2/2 + \gamma_1/6 \leq h + \delta
\]
yielding a coefficient $1.78 < 2.008$ according to (41) and (43).

3) $r_1, q_1$ and $(0, 0, 0)$. Here the average inequality reads

$$\epsilon_4 + 5\gamma_3/6 + (4\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.$$ 

Averaging with (67) gives

$$\epsilon_4 + 8\gamma_3/6 + 14\gamma_2/6 + \gamma_1/6 \leq h + \delta$$

yielding a coefficient $1.66 < 2.008$ according to (41) and (43).

4) $r_1, d_1, d_2$ and $(0, 0, 0)$. Here the average inequality reads

$$\epsilon_4 + 5\gamma_3/4 + (7\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta.$$ 

Averaging with (67) gives

$$\epsilon_4 + 9\gamma_3/8 + 11\gamma_2/4 + \gamma_1/8 \leq h + \delta$$

yielding a coefficient $1.92 < 2.008$ according to (41) and (43).

5) $r_1, q_1, d_1$ and $(0, 0, 0)$. Here the average inequality reads

$$\epsilon_4 + 6\gamma_3/4 + (6\gamma_2 - 2\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta.$$ 

Averaging with (67) gives

$$\epsilon_4 + 10\gamma_3/8 + 5\gamma_2/2 + \gamma_1/8 \leq h + \delta$$

yielding a coefficient $1.88 < 2.008$ according to (41) and (43).

6) $r_1, q_1, d_2$ and $(0, 0, 0)$. Here the average inequality reads

$$\epsilon_4 + 6\gamma_3/4 + (7\gamma_2 - 2\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta.$$ 

Averaging with (67) gives

$$\epsilon_4 + 10\gamma_3/8 + 21\gamma_2/8 + \gamma_1/8 \leq h + \delta$$

yielding a coefficient $1.80 < 2.008$ according to (41) and (43).

7) $r_1, q_1, d_1, d_2$ and $(0, 0, 0)$. Here the average inequality reads

$$\epsilon_4 + 7\gamma_3/5 + (9\gamma_2 - 2\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta.$$ 

Averaging with (67) gives

$$\epsilon_4 + 12\gamma_3/10 + 27\gamma_2/10 + \gamma_1/10 \leq h + \delta$$

yielding a coefficient $1.97 < 2.008$ according to (41) and (43).

Thus $d_4$ cannot occur in any list.
12 The case $s > 0$ in the main list

This case has caused a lot of trouble when I tried to exclude it. If $s > 0$ the main list does not end with the regular representation but with a transfer $(s_2, s_3, s)$ where the reduction of the constant term is not positive ie. $\kappa \leq 0$. This situation is rather odd and many of the strategies used up to now have to be abandoned. On the other hand side the set of possible transfers is much smaller than in the case $s = 0$ and this gives us the opportunity to examine all cases without the use of a computer. Here we go through some of the special features of the case $s > 0$. No optimal representation $n = x_4 a_4 + x_3 a_3 + x_2 a_2 + x_1 \leq n_h(A_4)$ can contain a second coefficient $x_2 \geq \gamma_2$, given that $x_4 \geq s$. Otherwhise we could apply the transfere $(s_2, s_3, s) \in C$ giving $n = (x_4 - s)a_4 + (x_3 + s\gamma_3)a_3 + (x_2 - \sigma)a_2 + (x_1 - \kappa)$. Here we write $\sigma$ for the reduction of the second term of $(s_2, s_3, s)$ and $\kappa$ for the ”reduction” of the constant term. Since $\kappa \leq 0$ and since we may assume $x_4 = \epsilon_4 - 1 > fh > s$ for some positive constant $f > 0$ all coefficients are nonnegative since $\sigma < \gamma_2$ and we arrived at a contradiction. Thus none of the transfers from $D^*$ or $E$ can give optimal representation as long as $s > 0$ since these transfers produce second terms $\geq \gamma_2$. That means that there are much fewer transfers available in the case $s > 0$ and therefore it is possible to present a complete consideration of all cases in all intervals without using computer results.

In the case $s > 0$ we experienced that many of the inequalities (41) had low values $b$ for the second coefficient in (41) and therefore the coefficient bounds for the $h$-range were bad. Thus it became neccessary to build a new routine to calculate the coefficient for the asymptotic $h$-range. Here is the only place where we consider numbers in the interval $[a_3, a_4]$. We look at

$$R = (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1 < a_4$$

This representation is regular. If it is also minimal we have

$$\gamma_3 + \gamma_2 + \gamma_1 \leq h + 5. \quad (68)$$

Of course this is much stronger than (24). If the representation of $R$ is not minimal we must have a transfer $(s_2, s_3, 0)$ of some elements $a_3$ in order to get the minimal representation. Of course $s_4 = 0$ since $R < a_4$. If $s_3 \geq 2$ we get

$$\gamma_3 + 3\gamma_2 \leq h + \delta, \quad (69)$$

where we left out the constant term. Now the last alternative is $s_3 = 1$. Here we get

$$R = (\gamma_3 - 3)a_3 + (2\gamma_2 - 2)a_2 + \gamma_1 - 1 - \beta_1^{(3)},$$

giving

$$\gamma_3 + 2\gamma_2 + \gamma_1 - \beta_1^{(3)} \leq h + \delta.$$
We now consider
\[ S = (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \beta_1^{(3)} - 1. \]
Of course here the transfer \((0, 1, 0)\) is impossible. If a transfer \((s_2, s_3, 0)\) with \(s_3 > 1\) would apply we would again have (69). Thus we have to consider the case where the regular representation of \(S\) is minimal, giving
\[ \gamma_3 + \gamma_2 + \beta_1^{(3)} \leq h + \delta. \]
Averaging the coefficient inequalities for \(R\) and \(S\) yields
\[ \gamma_3 + 3\gamma_2/2 + \gamma_1/2 \leq h + \delta, \tag{70} \]
We now combine our inequality (41)
\[ \epsilon_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta \]
with each of the new inequalities by non-negative weights \(x \geq 0, y \geq 0\) and \(z \geq 0\) and get
\[
\begin{align*}
\epsilon_4 + (a + x)\gamma_3 + (b + x)\gamma_2 + (c + x)\gamma_1 & \leq (1 + x)h + \delta \\
\epsilon_4 + (a + y)\gamma_3 + (b + 3y)\gamma_2 + c\gamma_1 & \leq (1 + y)h + \delta \\
\epsilon_4 + (a + z)\gamma_3 + (b + 3z/2)\gamma_2 + (c + z/2)\gamma_1 & \leq (1 + z)h + \delta 
\end{align*} \tag{71}
\]
Now we run through the interval \([0, 1]\) for the weights and register the least coefficient for the \(h\)-range for each of the three alternatives. The largest of these three coefficients then gives us an upper bound for the coefficient, since the cases (68), (69) and (70) cover all situations. In the case \(s > 0\) this method often gives much better coefficients than (41) and (43). Sometimes we mark a coefficient by an asterix \(*\) when we use this method to compute it.

12.1 Refined average method
The first runs on the computer showed that the cases where \(s > 0\) and \(r_s = 1\) were the most critical ones, giving large coefficient bounds for the \(h\)-range. Therefore we refined our method in this case. Now \(s > 0\) and \(r_s = 1\) mean that the transfer \((s_2, s_3, s)\) is used at the end of the list, and the transfer \((s_2 - 1, s_3, s)\) at another place. We now show that this has to be the first line. Suppose the transfer \((s_2 - 1, s_3, s)\) occurs in line \(f \neq 1\) then we have for the corresponding gains \(G(S^1) = G(s_2^{(1)}, s_3^{(1)}, s_4^{(1)}) > G(S^{(f)}) = G(s_2 - 1, s_3, s)\).
Then the transfer \((s_2^{(1)} + 1, s_3^{(1)}, s_4^{(1)})\) would be possible in the last line giving a larger gain than \((s_2, s_3, s)\), a contradiction.
Now we leave out the first line from the list and average. Since the transfers of the first and the last line only differ in the \(s_2\) position, with one unit, we have
\[ \kappa_1 = \kappa_l + \gamma_1, \text{ hence } \kappa_1 - \kappa_l = \gamma_1. \text{ Now } \sum_{i=2}^{l} s_4^{(i)} = \sum_{i=1}^{l} s_4^{(i)} - s \text{ and } \sum_{i=2}^{l} s_3^{(i)} = \sum_{i=1}^{l} s_3^{(i)} - |sw_p|. \text{ Thus averaging gives} \]

\[
\epsilon_4 + \left( 1 + \frac{\sum_{i=1}^{l} s_4^{(i)} - s}{l - 1} \right) \gamma_3 + \left( 1 + \frac{\sum_{i=1}^{l} s_3^{(i)} - |sw_p|}{l - 1} \right) \gamma_2 - \frac{\sum_{i=1}^{l} s_4^{(i)} - s}{l - 1} \beta_2^{(4)} + \frac{\gamma_1}{l - 1} \leq h + \delta. \quad (72)
\]

This inequality usually gives much better results than (47). If \( s > 0 \) and \( r_s = 0 \) we must have \( \kappa_l = 0 \). Otherwise we look at the transfer \((s_2 - 1, s_3, s)\) where the reduction of the constant term is \( 0 < \kappa_l + \gamma_1 < \gamma_1 \). The gain of this transfer \( G(s_2 - 1, s_3, s) \) is then \( \gamma_1 \) and this transfer is legal for \( N_1 \) but another transfer is optimal for \( N_1 \). Thus we can use the same argument as above and show that \((s_2, s_3, s)\) is not optimal at the end of the list.

We now go through all intervals from \( I_1 \) to \( I_{12} \) and show that the coefficient for the \( h \)-range is below 2.008 in all of them as long as \( s > 0 \) in the main list.

### 12.2 Interval \( I_1 \)

In this interval we only have to consider the main list:

\[ M(0) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1 \]

and try to find a representation for \( M(0) \) and all the numbers in the interval \( K(0) = [M(0) - \gamma_1 + 1, M(0)] \). First we check which transfers from the sets \( A, B, C \) or \( D \) can be used in the interval \( K(0) \). Also here it turns out that only very few transfers are possible.

In interval \( I_1 \) we have \( 0 < \beta_2^{(4)} < \frac{1}{6} \gamma_2 \). Therefore we have \( \gamma_2 > 6\beta_2^{(4)} > 4\beta_2^{(4)} > 2\beta_2^{(4)} > 0 \). Thus we are left with the possibilities \( r_1, r_2, r_3, r_4, r_5, r_6 \) and \( s = 1, 2, 3, 4, 5, 6 \). Remember that the case \( s = 0 \) is already settled.

Since \( \gamma_2 - \beta_2^{(4)} > \beta_2^{(4)} \) we see that \( s = 1 \) is impossible, since the corresponding transfer \((s_2, 0, 1) \in C \) could be used twice at the end of the list. \( s = 2 \) and \( s = 3 \) can be excluded by the same argument.

Only the second coefficient is reduced but after such a reduction there is still enough space to perform another one. So neither \( s = 1 \) nor \( s = 2 \) nor \( s = 3 \) can produce the optimal representation at the end of the list for \( K(0) \). We may even exclude \( s = 4, 5 \) when \( r_1 = 1 \) is used in the main list. If \( r_2 = 1 \) occurs in the main list we may exclude \( s = 4 \). No transfers from \( B \) can be used in the main list since \( s > 0 \).
Table 13 A. $K(0)$ for $s > 0$ in interval $I_1$, Part I, $r_6 = r_5 = 0$

<table>
<thead>
<tr>
<th>$r_6$</th>
<th>$r_5$</th>
<th>$r_4$</th>
<th>$r_3$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=4</td>
<td>0.6</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 6\gamma_3 + \gamma_2 - 5\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=5</td>
<td>0.78</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 7\gamma_3 + \gamma_2 - 6\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=6</td>
<td>0.96</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.04</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.04</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.11</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 10\gamma_3/2 + \gamma_2 - 8\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.04</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/2 + \gamma_2 - 9\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.18</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 12\gamma_3/2 + \gamma_2 - 10\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.39</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 12\gamma_3/2 + \gamma_2 - 10\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.40</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 14\gamma_3/2 + \gamma_2 - 12\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.40</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 16\gamma_3/2 + \gamma_2 - 14\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.62</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=4</td>
<td>0.6</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/2 + \gamma_2 - 9\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.18</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 14\gamma_3/2 + \gamma_2 - 11\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.25</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 17\gamma_3/2 + \gamma_2 - 13\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.40</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.04</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 15\gamma_3/2 + \gamma_2 - 12\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.42</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 18\gamma_3/2 + \gamma_2 - 14\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.44</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 21\gamma_3/2 + \gamma_2 - 16\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.75</td>
</tr>
</tbody>
</table>
Table 13 B. \(K(0)\) for \(s > 0\) in interval \(I_1\), Part II, \(r_6 = 0, r_5 = 1\)

<table>
<thead>
<tr>
<th>(r_6)</th>
<th>(r_5)</th>
<th>(r_4)</th>
<th>(r_3)</th>
<th>(r_2)</th>
<th>(r_1)</th>
<th>Average inequality</th>
<th>(s)</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\epsilon_4 + 11\gamma_3/2 + \gamma_2 - 9\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta) (s=4)</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>(\epsilon_4 + 15\gamma_3/3 + \gamma_2 - 12\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta) (s=6)</td>
<td>1.42</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>(\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta) (s=5)</td>
<td>1.04</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(\epsilon_4 + 18\gamma_3/4 + \gamma_2 - 14\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta) (s=6)</td>
<td>1.60</td>
<td></td>
</tr>
<tr>
<td>0</td>
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Table 13 C. $K(0)$ for $s > 0$ in interval $I_1$, Part III $r_6 = 1, r_5 = 0$

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In this interval we only have to consider the main list: $I$. Table 13 D. $K(0)$ for $s > 0$ in interval $I_1$, Part IV $r_6 = r_5 = 1$

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<td>$\epsilon_4 + 22 \gamma_3/4 + \gamma_2 - 18 \beta_2(4)/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=4$</td>
<td>1.63</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 22 \gamma_3/4 + \gamma_2 - 18 \beta_2(4)/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.63</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 22 \gamma_3/4 + \gamma_2 - 18 \beta_2(4)/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=6$</td>
<td>1.63</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 24 \gamma_3/5 + \gamma_2 - 19 \beta_2(4)/5 + \gamma_1/5 \leq h + \delta$</td>
<td>$s=6$</td>
<td>1.73</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 25 \gamma_3/5 + \gamma_2 - 20 \beta_2(4)/5 + \gamma_1/5 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.73</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 25 \gamma_3/5 + \gamma_2 - 20 \beta_2(4)/5 + \gamma_1/5 \leq h + \delta$</td>
<td>$s=6$</td>
<td>1.73</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 27 \gamma_3/6 + \gamma_2 - 21 \beta_2(4)/6 + \gamma_1/6 \leq h + \delta$</td>
<td>$s=6$</td>
<td>1.83</td>
</tr>
</tbody>
</table>

Thus there are no cases with coefficient over 2.008 in interval $I_1$.

### 12.3 Interval $I_2$

In this interval we only have to consider the main list:

$$M(0) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1$$
and try to find a representation for $M(0)$ and all the numbers in the interval $K(0) = [M(0) - \gamma_1 + 1, M(0)]$. First we check which transfers from the sets $A, B, C$ or $D$ can be used in the interval $K(0)$. Also here it turns out that only very few transfers are possible.

In interval $I_2$ we have $\frac{1}{6}\gamma_2 < \beta_2^{(4)} < \frac{1}{5}\gamma_2$. Therefore we have

$$\gamma_2 \times 5\beta_2^{(4)} > 4\beta_2^{(4)} > 3\beta_2^{(4)} > 2\beta_2^{(4)} > \beta_2^{(4)} > 6\beta_2^{(4)} - \gamma_2 > 0$$

Of course no transfers from $B$ or $D$ can be used since they would result in a second term $\geq \gamma_2$ which is not possible if $s > 0$. Now we assume that $(s_2, s_3, s_4) \in A$ is used. If $s_4 = 6$ we get

$$\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 6\beta_2^{(4)} \leq h + \delta.$$  

This contradicts (45) for $t = 6$ since $2\gamma_2 - 6\beta_2^{(4)} > \gamma_2/6$.

Thus we are left with the possibilities $r_1, r_2, r_3, r_4, r_5$ and $s = 1, 2, 3, 4, 5$

Remember that the case $s = 0$ is already settled.

Since $\gamma_2 - \beta_2^{(4)} > \beta_2^{(4)}$ we see that $s = 1$ is impossible, since the corresponding transfer $(s_2, 0, 1) \in C$ could be used twice at the end of the list. The case $s = 2$ can be excluded by the same argument.

Only the second coefficient is reduced but after such a reduction there is still enough space to perform another one. So neither $s = 1$ nor $s = 2$ can produce the optimal representation at the end of the list for $M(0)$. We may even exclude $s = 3$ when $r_1 = 1$ or $r_2 = 1$ in the main list. In the same way $s = 4$ is impossible when $r_1 = 1$ in the main list.
Table 14 A. $K(0)$ for $s > 0$ in interval $I_2$, Part I $r_5 = 0$

<table>
<thead>
<tr>
<th>$r_5$</th>
<th>$r_4$</th>
<th>$r_3$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=3$</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=4$</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 6\gamma_3 + \gamma_2 - 5\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 8\gamma_3/2 + \gamma_2 - 6\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.21</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 8\gamma_3/2 + \gamma_2 - 6\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=4$</td>
<td>1.21</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.31</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + \gamma_2 - 8\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.59</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=3$</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=3$</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=4$</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 10\gamma_3/2 + \gamma_2 - 8\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.42</td>
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<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 12\gamma_3/3 + \gamma_2 - 9\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.60</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 12\gamma_3/3 + \gamma_2 - 9\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=4$</td>
<td>1.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 13\gamma_3/3 + \gamma_2 - 10\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.64</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 15\gamma_3/4 + \gamma_2 - 11\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.84</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=3$</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=4$</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/2 + \gamma_2 - 9\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.53</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 13\gamma_3/3 + \gamma_2 - 10\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.64</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 13\gamma_3/3 + \gamma_2 - 10\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=4$</td>
<td>1.64</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 14\gamma_3/3 + \gamma_2 - 11\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.68</td>
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<td>1</td>
<td>$\epsilon_4 + 16\gamma_3/4 + \gamma_2 - 12\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.84</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=3$</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + \gamma_2 - 7\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=4$</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 15\gamma_3/3 + \gamma_2 - 12\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.73</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 17\gamma_3/4 + \gamma_2 - 13\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.85</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 12\gamma_3/3 + \gamma_2 - 9\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=4$</td>
<td>1.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 18\gamma_3/4 + \gamma_2 - 14\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=5$</td>
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</tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 20\gamma_3/5 + \gamma_2 - 15\beta_2^{(4)}/5 + \gamma_1/5 \leq h + \delta$</td>
<td>$s=5$</td>
<td>1.99</td>
</tr>
</tbody>
</table>
In this interval we have to consider different key numbers. We start with the main list:

$$M(0) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1$$
and try to find a representation for $M(0)$ and all the numbers in the inverall $K(0) = [M(0) - \gamma_1 + 1, M(0)]$. First we check which transfers from the sets $A, B, C$ or $D$ can be used in the interval $K(0)$. Also here it turns out that only very few transfers are possible.

In interval $I_3$ we have $\frac{1}{3} \gamma_2 < \beta_2^{(4)} < \frac{1}{4} \gamma_2$. Therefore we have

$\gamma_2 > 4 \beta_2^{(4)} > 3 \beta_2^{(4)} > 2 \beta_2^{(4)} > 6 \beta_2^{(4)} - \gamma_2 > \beta_2^{(4)} > 5 \beta_2^{(4)} - \gamma_2 > 0$

Of course no transfers from $B$ or $D$ can be used since they would result in a second term $\geq \gamma_2$ which is not possible if $s > 0$. Now we assume that $(s_2, s_3, s_4) \in A$ is used. If $s_4 = 5$ we get

$$\epsilon_4 + 6 \gamma_3 + 2 \gamma_2 - 5 \beta_2^{(4)} \leq h + \delta.$$ 

This contradicts (45) for $t = 4$ since $2 \gamma_2 - 5 \beta_2^{(4)} > \gamma_2/3$. For $s_4 = 6$ we have

$$\epsilon_4 + 7 \gamma_3 + 2 \gamma_2 - 6 \beta_2^{(4)} \leq h + \delta.$$ 

This contradicts (45) for $t = 6$ since $2 \gamma_2 - 6 \beta_2^{(4)} > \gamma_2/6$.

Thus we are left with the possibilities $r_1, r_2, r_3, r_4$ and $s = 1, 2, 3, 4$. Remember that the case $s = 0$ is already settled. All these possibilities are listed up in the table below. Only two cases have coefficients above 2.008.

Since $\gamma_2 - \beta_2^{(4)} > \beta_2^{(4)}$ we see that $s = 1$ is impossible, since the corresponding transfer $(s_2, 0, 1) \in C$ could be used twice at the end of the list. The case $s = 2$ can be excluded by the same argument. Only the second coefficient is reduced but after such a reduction there is still enough space to perform another one.

So neither $s = 1$ nor $s = 2$ can produce the optimal representation at the end of the list for $K(0)$. We may even exclude $s = 3$ when $r_1 = 1$ in the main list.
Thus there are only two cases left: $K(0)$ is represented by $r_1, r_2, r_3$ and $s = 4$ or
$r_1, r_2, r_3, r_4$ and $s = 4$.

Now we turn to the different orderings of the main list.

The transferes $(0, 0, 1), (0, 2), 0, 2)$ and $(0, 3), 0, 3)$ and maybe $(0, 4), 0, 4)$ are
used. At the end of the list we have $s = 4$. Here we get 6 orderings. If
$(0, 4), 0, 4)$ enters the main list this happens at the very beginning of the list.

Since we never make use of the first lines of our list it is sufficient to consider
the case when $(0, 4), 0, 4)$ does not enter the main list.

A) $r_2, r_1, r_3$ and $s = 4$

\[ \epsilon_4 + 3 \gamma_3 + \gamma_2 - 2 \beta_2^{(4)} + \gamma_1 - 2 \beta_1^{(4)} + [0, 2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} + [0, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} - [0, 3]\gamma_1 - 4\beta_1^{(4)} + [0, 4]\gamma_1 \leq h + \delta. \]

If we use the weights 0, 1, 0 and 1 this gives
\[ \epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta, \]
yielding a coefficient 1.46 < 2.008 according to (41) and (43).

B) \( r_3, r_1, r_2 \) and \( s = 4 \)
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} + [0, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} - [0, 3]\gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} + [0, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - 4\beta_1^{(4)} + [0, 4]\gamma_1 \leq h + \delta. \]

If we use the weights 0, 1, 0 and 1 this gives
\[ \epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta, \]
yielding a coefficient 1.46 < 2.008 according to (41) and (43).

C) \( r_1, r_3, r_2 \) and \( s = 4 \)
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} + [0, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} - [0, 3]\gamma_1 - 2\beta_1^{(4)} + [0, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - 4\beta_1^{(4)} + [0, 4]\gamma_1 \leq h + \delta. \]

Since \(-4\beta_1^{(4)} + [0, 4]\gamma_1 \geq 0\) because of the final line, this line gives us the following information:
\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 \leq h + \delta. \]

Averaging with line two in the system gives
\[ \epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
yielding a coefficient 1.79 < 2.008 according to (41) and (43).

D) \( r_2, r_3, r_1 \) and \( s = 4 \)
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} + [0, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - 3\beta_1^{(4)} + [0, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + [0, 3]\gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} + [0, 4]\gamma_1 \leq h + \delta. \]
Since $-4\beta_1^{(4)} + [0, 4] \gamma_1 \geq 0$ because of the final line, this line gives us the following information:

$$\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.$$ 

Averaging with line two in the system gives

$$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$$

yielding a coefficient 1.79 < 2.008 according to (41) and (43). The following two inequality systems cannot be excluded easily.

E) $r_1, r_2, r_3$ and $s = 4$

$$\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_1^{(4)} + \beta_1^{(3)} - 2\beta_1^{(4)} + [0, 2] \gamma_1 &\leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2] \gamma_1 - 3\beta_1^{(4)} + [0, 3] \gamma_1 &\leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} - [0, 3] \gamma_1 - 4\beta_1^{(4)} + [0, 4] \gamma_1 &\leq h + \delta.
\end{align*}$$

We now switch to $K(2)$ and compare the list with the main list. Since $r_1$ is used first in the main list this must be the case for $K(2)$, too. There we get

$$\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta$$

We now exchange this line with line number two from the main list, namely

$$\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + [0, 2] \gamma_1 - \beta_1^{(4)} \leq h + \delta.$$ 

Here we must have $[0, 2] = 1$. Thus we get the additional contribution $\gamma_3 + \gamma_2 - 3\beta_2^{(4)}$, which will help us to exclude the majority of all cases.

F) $r_3, r_2, r_1$ and $s = 4$

$$\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} + [0, 3] \gamma_1 &\leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_1^{(4)} + 3\beta_1^{(3)} - [0, 3] \gamma_2 - 2\beta_1^{(4)} + [0, 2] \gamma_1 &\leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2] \gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} + [0, 4] \gamma_1 &\leq h + \delta.
\end{align*}$$

Here $r_1$ is used at the end of the list. This must be the case for $K(2)$, too. There we get

$$\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.$$
for the final line. We now exchange this line with line number three from the main list, namely
\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} - [0, 2]\gamma_1 \leq h + \delta.
\]
Here we must have \([0, 2] = 0\). Thus we get the same additional contribution \(\gamma_3 + \gamma_2 - 3\beta_2^{(4)}\) as in case E). This will help us to exclude the majority of all cases.

We now jump over \(M(4)\) and \(M(3)\) and consider
\[
M(2) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (2\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1
\]
and try to find a representation for \(M(2)\) and all the numbers in the inverall \(K(2) = [M(2) - \gamma_1 + 1, M(2)]\). First we check which transfers from the sets \(A, B^*, C, D^*\) or \(E\) can be used in the interval \(K(2)\). Also here it turns out that only very few transfers are possible.

Of course no transfers from \(D^*\) or \(E\) can be used since they would result in a second term \(\geq \gamma_2\) which is not possible if \(s > 0\). Now we assume that \((s_2, s_3, s_4) \in A\) is used. We know that \(1 < s_4 \leq 4\) is impossible. But \(s_4 = 1, s_4 = 5\) and \(s_4 = 6\) are possible. This means also that \(s = 0, s = 1, s = 5\) and \(s = 6\) are possible at the end of the list for \(K(2)\). Now if \(s = 1\) represents a transfer with positive gain then it must be possible to perform it in addition to \(r_1\) in the main list since \(\gamma_2 - \beta_2^{(4)} > \beta_2^{(4)}\). In the same manner we can exclude \(s = 5\) and \(s = 6\) since \(\gamma_2 - \beta_2^{(4)} > 5\beta_2^{(4)} - \gamma_2\) and \(\gamma_2 - \beta_2^{(4)} > 6\beta_2^{(4)} - \gamma_2\). Thus \(s = 0\) for \(K(2)\).

Now we assume that \((s_2, s_3, s_4) \in B^*\) is used. If \(s_4 \leq 2\) we get a second term \(\geq \gamma_2\). If \(s_4 \geq 4\) we have
\[
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} \leq h + \delta.
\]
This contradicts \((45)\) for \(t = 4\) since \(\gamma_2 - 2\beta_2^{(4)} > \gamma_2/2\).

Since \(r_1\) is used in the mainlist we know it has to be used in the list for \(K(2)\), too. Thus we get eight possibilities for \(K(2)\):
Table 16. \(K(2)\) for \(s = 0\) and \(s > 0\) in the main list in interval \(I_3\)

<table>
<thead>
<tr>
<th>(q_3)</th>
<th>(r_6)</th>
<th>(r_5)</th>
<th>(r_1)</th>
<th>Average inequality</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>(\epsilon_4 + 3\gamma_3/2 + 3\beta^{(4)}_2/2 + \gamma_1/2 \leq h + \delta)</td>
<td>3.2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(\epsilon_4 + 9\gamma_3/3 + \gamma_2/3 + \gamma_3/3 \leq h + \delta) with additonal contribution (\gamma_3 + \gamma_2 - 3\beta^{(4)}_2)</td>
<td>2.26</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - 3\beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta)</td>
<td>1.91</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>(\epsilon_4 + 10\gamma_3/3 + (\gamma_2 - \beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta) with additonal contribution (\gamma_3 + \gamma_2 - 3\beta^{(4)}_2)</td>
<td>2.26</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 4\beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta)</td>
<td>1.89</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(\epsilon_4 + 16\gamma_3/4 + (2\gamma_2 - 4\beta^{(4)}_2)/4 + \gamma_1/4 \leq h + \delta) with additonal contribution (\gamma_3 + \gamma_2 - 3\beta^{(4)}_2)</td>
<td>2.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\epsilon_4 + 17\gamma_3/4 + (3\gamma_2 - 7\beta^{(4)}_2)/4 + \gamma_1/4 \leq h + \delta)</td>
<td>1.98</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>(\epsilon_4 + 7\gamma_3/3 + (\gamma_2 + 2\beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta) with additonal contribution (\gamma_3 + \gamma_2 - 3\beta^{(4)}_2)</td>
<td>2.44</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\epsilon_4 + 8\gamma_3/3 + (2\gamma_2 - \beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta)</td>
<td>1.98</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(\epsilon_4 + 13\gamma_3/4 + (2\gamma_2 - \beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta) with additonal contribution (\gamma_3 + \gamma_2 - 3\beta^{(4)}_2)</td>
<td>2.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\epsilon_4 + 14\gamma_3/4 + (3\gamma_2 - 4\beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta)</td>
<td>1.85</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>(\epsilon_4 + 14\gamma_3/4 + (2\gamma_2 - 2\beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta) with additonal contribution (\gamma_3 + \gamma_2 - 3\beta^{(4)}_2)</td>
<td>2.17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\epsilon_4 + 15\gamma_3/4 + (3\gamma_2 - 5\beta^{(4)}_2)/3 + \gamma_1/3 \leq h + \delta)</td>
<td>1.87</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(\epsilon_4 + 20\gamma_3/5 + (3\gamma_2 - 5\beta^{(4)}_2)/5 + \gamma_1/5 \leq h + \delta) with additonal contribution (\gamma_3 + \gamma_2 - 3\beta^{(4)}_2)</td>
<td>2.12</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\epsilon_4 + 21\gamma_3/5 + (4\gamma_2 - 8\beta^{(4)}_2)/5 + \gamma_1/5 \leq h + \delta)</td>
<td>1.90</td>
</tr>
</tbody>
</table>

Thus we are left with one single case where \(r_1\) is used in addition to the regular representation, giving

\[
\epsilon_4 + 2\gamma_3 + \beta^{(4)}_2 + \gamma_1 - \beta^{(4)}_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 2\beta^{(4)}_2 + \beta^{(4)}_1 \leq h + \delta.
\]

In case E) the final line in the main list reads:

\[
\epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta^{(4)}_2 + 3\beta^{(4)}_1 - [0, 3]\gamma_1 - 4\beta^{(4)}_1 + [0, 4]\gamma_1 \leq h + \delta.
\]

Since \(- [0, 3] + [0, 4]\gamma_1\) has to be 1 we can combine this line with line 2 for \(K(2)\) giving:
\[ \epsilon_4 + 6\gamma_3/2 + (\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_2/2 \leq h + \delta. \]

This yields a coefficient 1.92 by (71).

In case F) the final line in the main list reads:

\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} + [0, 4] \gamma_1 \leq h + \delta. \]

Now \(-4\beta_1^{(4)} + [0, 4] \gamma_1 \geq 0\) since this is the final line in the list and therefore we have

\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta. \]

We can combine this line with line 1 for \(K(2)\) giving:

\[ \epsilon_4 + 7\gamma_3/2 + (\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_2/2 \leq h + \delta. \]

This yields a coefficient 2.004 by (71).

Thus we finished interval \(I_3\).

### 12.5 Interval \(I_4\)

In this interval we have to consider different key numbers. We start with the main list:

\[ M(0) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1 \]

and try to find a representation for \(M(0)\) and all the numbers in the interval \(K(0) = [M(0) - \gamma_1 + 1, M(0)]\). First we check which transfers from the sets \(A, B, C\) or \(D\) can be used in the interval \(K(0)\). Also here it turns out that only very few transfers are possible.

In interval \(I_4\) we have \(\frac{1}{3}\gamma_2 < \beta_2^{(4)} < \frac{1}{3}\gamma_2\). Therefore we have

\[ \gamma_2 > 3\beta_2^{(4)} > 6\beta_2^{(4)} - \gamma_2 > 2\beta_2^{(4)} > 5\beta_2^{(4)} - \gamma_2 > \beta_2^{(4)} > 4\beta_2^{(4)} - \gamma_2 > 0 \]

Of course no transfers from \(B\) or \(D\) can be used since they would result in a second term \(\geq \gamma_2\) which is not possible if \(s > 0\). Now we assume that \((s_2, s_3, s_4) \in A\) is used. If \(s_4 = 5\) we get

\[ \epsilon_4 + 6\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} \leq h + \delta. \]

This contradicts (45) for \(t = 4\) since \(2\gamma_2 - 5\beta_2^{(4)} > \gamma_2/3\). For \(s_4 = 4\) we have

\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 4\beta_2^{(4)} \leq h + \delta. \]

This contradicts (45) for \(t = 4\) since

\[ 2\gamma_2 - 4\beta_2^{(4)} > \gamma_2/2. \]

Thus we are left with the possibilities \(r_1, r_2, r_3, r_6\) and \(s = 1, 2, 3, 6\). Remember that the case \(s = 0\) is already settled.

Since \(\gamma_2 - \beta_2^{(4)} > \beta_2^{(4)}\) we see that \(s = 1\) is impossible, since the corresponding transfer \((s_2, 0, 1) \in C\) could be used twice at the end of the list. We may even exclude \(s = 2\) when \(r_1 = 1\) in the main list.
<table>
<thead>
<tr>
<th>$r_6$</th>
<th>$r_3$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=2</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=3</td>
<td>1.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 6\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=6</td>
<td>0.96</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 6\gamma_3/2 + (2\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=3</td>
<td>1.85</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (3\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.23</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=2</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=3</td>
<td>1.89*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 10\gamma_3/2 + (3\gamma_2 - 8\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.50</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=3</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/3 + (4\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.77</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.89*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=3</td>
<td>1.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/2 + (3\gamma_2 - 9\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.80</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 6\gamma_3/2 + (2\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=3</td>
<td>1.85</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 13\gamma_3/3 + (4\gamma_2 - 10\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.93</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.89*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=3</td>
<td>1.89*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 14\gamma_3/3 + (4\gamma_2 - 11\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.94*</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=3</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/4 + (5\gamma_2 - 12\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=6</td>
<td>2.25</td>
</tr>
</tbody>
</table>
Table 17 B. $K(0)$ for $s > 0$ in interval $I_4$, Part II $r_6 = 1$

<table>
<thead>
<tr>
<th>$r_6$</th>
<th>$r_3$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$e_4 + 10\gamma_3/2 + (3\gamma_2 - 8\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 11\gamma_3/2 + (3\gamma_2 - 9\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=3</td>
<td>1.80</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 7\gamma_3 + 2\gamma_2 - 6\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=6</td>
<td>0.96</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$e_4 + 13\gamma_3/3 + (4\gamma_2 - 10\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=3</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 9\gamma_3/2 + (3\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.23</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$e_4 + 10\gamma_3/2 + (3\gamma_2 - 8\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 14\gamma_3/3 + (4\gamma_2 - 11\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=3</td>
<td>1.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 10\gamma_3/2 + (3\gamma_2 - 8\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.50</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$e_4 + 16\gamma_3/4 + (5\gamma_2 - 12\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=3</td>
<td>2.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 12\gamma_3/3 + (4\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.77</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$e_4 + 14\gamma_3/3 + (4\gamma_2 - 11\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=2</td>
<td>1.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=3</td>
<td>1.89</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 7\gamma_3/2 + (2\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.89</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$e_4 + 13\gamma_3/3 + (4\gamma_2 - 10\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=3</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 13\gamma_3/3 + (4\gamma_2 - 10\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.93</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$e_4 + 14\gamma_3/3 + (4\gamma_2 - 11\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=2</td>
<td>1.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 14\gamma_3/3 + (4\gamma_2 - 11\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=3</td>
<td>1.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 14\gamma_3/3 + (4\gamma_2 - 11\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.94</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$e_4 + 16\gamma_3/4 + (5\gamma_2 - 12\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=3</td>
<td>2.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$e_4 + 16\gamma_3/4 + (5\gamma_2 - 12\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=6</td>
<td>2.25</td>
</tr>
</tbody>
</table>

We now show that $r_6 = 0$ and that $s = 6$ is impossible in the main list. We assume that $r_6 = 1$ or $s = 6$ is used in the main list. This gives $e_4 + 7\gamma_3 + 2\gamma_2 - 6\beta_2^{(4)} \leq h + \delta$ in the main list and we may look at

$$M(6) = (e_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (6\beta_2^{(4)} - \gamma_2 - 1)a_2 + \gamma_1 - 1$$

and try to find a representation for $M(6)$ and all the numbers in the inverall $K(6) = [M(6) - \gamma_1 + 1, M(6)]$. In all cases with a coefficient above 2.008 in the above list we have $r_1 = 1$. Thus we cannot have $s = 1$ in the list for $K(6)$.

Otherwise the transfer $s = 1$ could be used in addition to $r_1$ in the main list. If $s = 4$ in the $K(6)$-list we get $e_4 + 5\gamma_3 + 2\beta_2^{(4)} \leq h + \delta$. This contradicts (45) for $t = 4$ since $2\beta_2^{(4)} > \gamma_2/2$. If $s = 5$ in the $K(6)$-list we get

$$e_4 + 6\gamma_3 + \beta_2^{(4)} \leq h + \delta.$$

Averaging this inequality with $e_4 + 7\gamma_3 + 2\gamma_2 - 6\beta_2^{(4)} \leq h + \delta$ using the weights 6 and 1 gives
yielding a coefficient 1.96 < 2.008 according to (41) and (43). Since s = 3 or s = 6 would give a negative second coefficient we are left with s = 0 for the K(6)-list. But then we may argue as in the chapter where we excluded r_6 = 1 from all main lists by showing that there is no way to represent K(6). Thus we are left with two cases for the main list in $I_4$: $r_1, r_2, r_3$ and $s = 3$ in addition to $r_1, r_2$ and $s = 3$. We now proceed to

$$M(3) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (3\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1$$

and try to find a representation for $M(3)$ and all the numbers in the interval $K(3) = [M(3) - \gamma_1 + 1, M(3)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(3)$. Also here it turns out that only very few transfers are possible. No transfers from $D^*$ or $E$ can give optimal representations as long as $s > 0$ since these transfers produce second terms larger than $\gamma_2$. Only $r_1, r_2$ and $r_6$ from $A$ may be used. If $r_4$ is used we get

$$\epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} \leq h + \delta.$$ 

This contradicts (45) for $t = 4$ since $\gamma_2 - \beta_2^{(4)} > \gamma_2/2$. If $r_5$ is used we get

$$\epsilon_4 + 6\gamma_3 + \gamma_2 - 2\beta_2^{(4)} \leq h + \delta.$$ 

This contradicts (45) for $t = 5$ since $\gamma_2 - 2\beta_2^{(4)} > \gamma_2/3$. Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \leq 3$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$. If $s_4 > 3$ we get

$$\epsilon_4 + 5\gamma_3 + 3\beta_2^{(4)} \leq h + \delta.$$ 

This contradicts (45) for $t = 4$ since $3\beta_2^{(4)} > \gamma_2/2$. Thus we are left with $r_1, r_2, r_6$ and $s = 0, 2$ or $s = 6$ for $K(3)$. We know $r_1$ and $r_2$ have to occur. The possibility $s = 1$ is excluded since the corresponding transfer could be used twice at the end of the list.

### Table 18. $K(3)$ for $s > 0$ in the main list in interval $I_4$

<table>
<thead>
<tr>
<th>$r_6$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 6\gamma_3/3 + 6\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.77</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3/2 + 3\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.89*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.77</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 13\gamma_3/4 + (\gamma_2 + 3\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>2.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=2</td>
<td>1.77</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.77</td>
</tr>
</tbody>
</table>
From the main list we get an additional contribution \( \gamma_2 - 3\beta_2^{(4)} \). This turns the inequality \( \epsilon_4 + 13\gamma_3/4 + (\gamma_2 + 3\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \) from the last case with coefficient bound above 2.008 into \( \epsilon_4 + 13\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta \) with a coefficient 1.99 < 2.008 according to (41) and (43).

Thus we have two cases for the main list \((r_3), r_2, r_1\) and \(s = 3\) and one case for \(K(3)\) namely \(r_2, r_1\) and \(s = 0\).

We now study

\[
M(2) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (2\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1
\]

and try to find a representation for \(M(2)\) and all the numbers in the interval \(K(2) = [M(2) - \gamma_1 + 1, M(2)]\). First we check which transfers from the sets \(A, B^*, C, D^*\) or \(E\) can be used in the interval \(K(2)\). Also here it turns out that only very few transfers are possible. No transfers from \(D^*\) or \(E\) can give optimal representaions as long as \(s > 0\) since these transfers produce second terms larger than \(\gamma_2\).

Only \(r_1, r_4\) and \(r_5\) from \(A\) may be used. We also know \(s = 0\) since this is the case for \(K(3)\).

Now we assume that \((s_2, s_3, s_4) \in B^*\) is used. If \(s_4 \leq 2\) the second term will be \(\geq \gamma_2\) which is impossible since \(s > 0\). If \(s_4 > 3\) we get

\[
\epsilon_4 + 5\gamma_3 + 2\beta_2^{(4)} \leq h + \delta.
\]

This contradicts (45) for \(t = 4\) since \(2\beta_2^{(4)} > \gamma_2/2\). Thus we are left with \(r_1, r_4, r_5, q_3\) and \(s = 0\) for \(K(2)\). We know \(r_1\) has to occur.

### Table 19. \(K(2)\) for \(s > 0\) in the main list in interval \(I_4\)

<table>
<thead>
<tr>
<th>(q_3)</th>
<th>(r_5)</th>
<th>(r_4)</th>
<th>(r_1)</th>
<th>Average inequality</th>
<th>(s)</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 1</td>
<td>(\epsilon_4 + 3\gamma_3/2 + 3\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta)</td>
<td>s=0</td>
<td>3.41</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0 1 1</td>
<td>(\epsilon_4 + 8\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta)</td>
<td>s=0</td>
<td>2.35</td>
<td>b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 1 0 1</td>
<td>(\epsilon_4 + 9\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta)</td>
<td>s=0</td>
<td>2.37</td>
<td>c</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 1 1 1</td>
<td>(\epsilon_4 + 13\gamma_3/4 + (2\gamma_2 - 2\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta)</td>
<td>s=0</td>
<td>2.30</td>
<td>d</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 0 0 1</td>
<td>(\epsilon_4 + 7\gamma_3/3 + (\gamma_2 + 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta)</td>
<td>s=0</td>
<td>2.38</td>
<td>e</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 0 1 1</td>
<td>(\epsilon_4 + 12\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta)</td>
<td>s=0</td>
<td>2.16</td>
<td>f</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 1 0 1</td>
<td>(\epsilon_4 + 13\gamma_3/4 + (2\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta)</td>
<td>s=0</td>
<td>2.22</td>
<td>g</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 1 1 1</td>
<td>(\epsilon_4 + 18\gamma_3/5 + (3\gamma_2 - 3\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta)</td>
<td>s=0</td>
<td>2.22</td>
<td>h</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This gives us 8 cases to consider.

a) \(K(2)\) is represented by \(r_1\) and \(s = 0\).

The list looks like this

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta
\end{align*}
\]
If $r_1$ comes before $r_2$ in the main list we get there
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta. \]
Together with the last line of the list for $K(2)$ this gives
\[ \epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta \]
with a coefficient $2.0 < 2.008$ according to (41) and (43).
If $r_2$ comes before $r_1$ in the main list we get there
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta. \]
Together with the first line of the list for $K(2)$ this gives the same inequality and the same coefficient bound as above.

b) $K(2)$ is represented by $r_1, r_4$ and $s = 0$.
If $r_1$ comes before $r_2$ in the main list we get there
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta. \]
We compare this with the first line of the list for $K(2)$
\[ \epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta. \]
Here we get an additional contribution, the amount of which is $\gamma_3 + \gamma_2 - 3\beta_2^{(4)}$ and the average inequality
\[ \epsilon_4 + 8\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta. \]
turns into
\[ \epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta. \]
with a coefficient $2.0019 < 2.008$ by (71).
If $r_2$ comes before $r_1$ in the main list we get there
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta. \]
Again we compare this with the list for $K(2)$, this time the last line
\[ \epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta. \]
Here we get the same additional contribution $\gamma_3 + \gamma_2 - 3\beta_2^{(4)}$, the same average inequality and the same coefficient bound as above.
c) $K(2)$ is represented by $r_1, r_5$ and $s = 0$. Here the additional contribution is not enough to exclude the case so this case has to wait until the end.
d) $K(2)$ is represented by $r_1, r_4, r_5$ and $s = 0$. Here we go through the different orderings for the list. The transfer $r_1$ has to be at the beginning or at the end.
of the list, because the list for \( K(3) \) consists only of \( r_1, r_2 \) and \( s = 0 \). Therefore \( r_4 \) and \( r_5 \) will follow each other and we expect to get an inequality with one single \( \beta_1^{(4)} \).

\( \alpha \) \( r_1, r_5, r_4 \) and \( s=0 \).

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - \beta_1^{(3)} + [5,1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [5,1]\gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [4,1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [4,1]\gamma_1 & \leq h + \delta.
\end{align*}
\]

The weight distribution 1, 0, 1 and 0 gives

\[
\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
yielding a coefficient 1.58 < 2.008 according to (41) and (43).

\( \beta \) \( r_1, r_4, r_5 \) and \( s=0 \).

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - \beta_1^{(3)} + [4,1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [4,1]\gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [5,1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [5,1]\gamma_1 & \leq h + \delta.
\end{align*}
\]

The weight distribution 0, 1, 2 and 1 gives

\[
\epsilon_4 + 18\gamma_3/4 + (3\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
\]
yielding a coefficient 2.00 < 2.008 according to (41) and (43).

\( \gamma \) \( r_4, r_5, r_1 \) and \( s=0 \).

\[
\begin{align*}
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [4,1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [4,1]\gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [5,1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [5,1]\gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]

The weight distribution 0, 1, 0, and 1 gives

\[
\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
yielding a coefficient 1.58 < 2.008 according to (41) and (43).
\( \text{\( \delta \)} \) \( r_5, r_4, r_1 \) and \( s=0 \).

\[
\begin{align*}
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [5, 1] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [5, 1] \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [4, 1] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [4, 1] \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta.
\end{align*}
\]

The weight distribution 1, 2, 1 and 0 gives

\[
\epsilon_4 + 18\gamma_3/4 + (3\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
\]

yielding a coefficient 2.00 < 2.008 according to (41) and (43).

e) \( K(2) \) is represented by \( r_1, q_3 \) and \( s = 0 \). Like in case b) we can get an additional contribution \( \gamma_3 + \gamma_2 - 3\beta_2^{(4)} \). Therefore the inequality \( \epsilon_4 + 7\gamma_3/3 + (\gamma_2 + 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \) turns into

\[
\epsilon_4 + 8\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \text{ yielding a coefficient } 1.92 < 2.008 \text{ according to (41) and (43).}
\]

f) \( K(2) \) is represented by \( r_1, r_4, q_3 \) and \( s = 0 \). From the list for \( K(3) \) we get an additional contribution \( \beta_2^{(4)} \). Therefore the inequality \( \epsilon_4 + 12\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta \) turns into

\[
\epsilon_4 + 12\gamma_3/4 + (2\gamma_2 + \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \text{ yielding a coefficient } 2.0028 < 2.008 \text{ according to (41) and (43).}
\]

g) \( K(2) \) is represented by \( r_1, r_5, q_3 \) and \( s = 0 \). From the list for \( K(3) \) we get an additional contribution \( \beta_2^{(4)} \). Therefore the inequality \( \epsilon_4 + 13\gamma_3/4 + (2\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \) turns into

\[
\epsilon_4 + 13\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta \text{ yielding a coefficient } 1.99 < 2.008 \text{ according to (41) and (43).}
\]

h) \( K(2) \) is represented by \( r_1, r_4, r_5, q_3 \) and \( s = 0 \).

A) \( r_1 \) comes first in the list for \( K(3) \) and there we get

\[
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta
\]

In the list for \( K(2) \) the transfer \( r_1 \) also comes first.

\( \alpha) \) \( r_5 \) comes just before \( r_4 \) in the \( K(2) \)-list. This gives

\[
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5] \gamma_1 - 4\beta_1^{(4)} + \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta.
\]

Together with the first line of the \( K(3) \)-list this yields

\[
\epsilon_4 + 7\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]

yielding a coefficient 1.13 < 2.008 according to (41) and (43).

\( \beta) \) \( r_5 \) comes just before \( q_3 \) in the \( M(2) \)-list. This gives
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1,5]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3]\gamma_1 \leq h + \delta. \]

Averaging this line and the first line of the \( K(3) \)-list using the weight distribution 2 and 1 yields
\[ \epsilon_4 + 8\gamma_3/3 + (\gamma_2 + 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.84 < 2.008 according to (41) and (43).

\( r_4 \) comes just before \( q_3 \) in the \( M(2) \)-list. This gives
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1,4]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3]\gamma_1 \leq h + \delta. \]

Together with the first line of the \( K(3) \)-list this yields
\[ \epsilon_4 + 6\gamma_3/2 + (\gamma_2 + \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
yielding a coefficient 1.06 < 2.008 according to (41) and (43).

\( r_1, q_3, r_4, r_5 \) and \( s = 0 \)
\[ \epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} + [1,3]\gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1,4]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1,4]\gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [1,5]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1,5]\gamma_1 \leq h + \delta. \]

The weight distribution 0, 1, 2, 1 and 1 gives
\[ \epsilon_4 + 21\gamma_3/5 + (4\gamma_2 - 6\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta \]
yielding a coefficient 1.90 < 2.008 according to (41) and (43).

B) \( r_1 \) comes last in the list for \( K(3) \) and there we get
\[ \epsilon_4 + \gamma_3 + 3\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta \]

In the list for \( K(2) \) the transfer \( r_1 \) also comes last.

\( r_4 \) comes just before \( r_5 \) in the \( M(2) \)-list. This gives
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1,4]\gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [1,5]\gamma_1 \leq h + \delta. \]

Together with the last line of the \( K(3) \)-list this yields
\[ \epsilon_4 + 7\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta \]
yielding a coefficient $1.13 < 2.008$ according to (41) and (43).

$\beta$) $q_3$ comes just before $r_5$ in the $M(2)$-list. This gives

$$\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5]\gamma_1 \leq h + \delta.$$ 

Averaging this line and the last line of the $K(3)$-list yields using the weight distribution 2 and 1 yields

$$\epsilon_4 + 8\gamma_3/3 + (\gamma_2 + 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$$

giving a coefficient $1.84 < 2.008$ according to (41) and (43).

$\gamma$) $q_3$ comes just before $r_4$ in the $M(2)$-list. This gives

$$\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4]\gamma_1 \leq h + \delta.$$ 

Together with the last line of the $K(3)$-list this yields

$$\epsilon_4 + 6\gamma_3/2 + (\gamma_2 + \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$$

yielding a coefficient $1.06 < 2.008$ according to (41) and (43).

$\delta$) $r_5, r_4, q_3, r_1$ and $s = 0$

$$\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5]\gamma_1 \leq h + \delta$$

$$\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5]\gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4]\gamma_1 \leq h + \delta$$

$$\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta$$

$$\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - \beta_1^{(4)} \leq h + \delta$$

$$\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.$$ 

The weight distribution 1, 1, 2, 1 and 0 gives

$$\epsilon_4 + 21\gamma_3/5 + (4\gamma_2 - 6\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta$$

yielding a coefficient $1.90 < 2.008$ according to (41) and (43).

Thus we are left with alternative c) where $K(2)$ is represented by $r_5, r_1$ and $s = 0$. First we summarize what we know and present all inequalities.

A) If $r_1$ comes before $r_2$ we get the following main list

$$\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} + [0, 3]\gamma_1 \leq h + \delta$$

$$\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} - [0, 3]\gamma_1 - \beta_1^{(4)} \leq h + \delta$$

$$\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} + [0, 2]\gamma_1 \leq h + \delta$$

$$\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - 3\beta_1^{(4)} + [0, 3]\gamma_1 \leq h + \delta$$
or
\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} + [0, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - 3\beta_1^{(4)} + [0, 3]\gamma_1 & \leq h + \delta
\end{align*}
\]

For $K(3)$ we have
\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} + [0, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 & \leq h + \delta
\end{align*}
\]

For $K(2)$ we have
\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 50]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5]\gamma_1 & \leq h + \delta
\end{align*}
\]

B) If $r_2$ comes before $r_1$ we get the following main list
\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} + [0, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} - [0, 3]\gamma_1 - 2\beta_1^{(4)} + [0, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} + [0, 3]\gamma_1 & \leq h + \delta
\end{align*}
\]

or
\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} + [0, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} + [0, 3]\gamma_1 & \leq h + \delta
\end{align*}
\]

For $K(3)$ we have
\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} + [0, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} + 2\beta_1^{(4)} - [0, 2]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta
\end{align*}
\]

For $K(2)$ we have
\[
\begin{align*}
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5]\gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta
\end{align*}
\]
Now we have to consider

\[ M(5) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (5\beta_2^{(4)} - \gamma_2 - 1)a_2 + \gamma_1 - 1 \]

and try to find a representation for \( M(5) \) and all the numbers in the interval \( K(5) = [M(5) - \gamma_1 + 1, M(5)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(5) \). Also here it turns out that only very few transfers are possible. No transfers from \( D^* \) or \( E \) can give optimal representaions as long as \( s > 0 \) since these transferes produce second terms larger than \( \gamma_2 \).

Only \( r_1 \) and \( r_4 \) from \( A \) may be used.

Now we assume that \( (s_2, s_3, s_4) \in B^* \) is used. If \( s_4 = 1, 4, 5 \) or \( s = 6 \) the second term will be \( \geq \gamma_2 \) which is impossible since \( s > 0 \). Thus we are left with \( r_1, r_4, q_2, q_3 \) and \( s = 0 \). We know \( r_1 \) is used.

### Table 21. \( K(5) \) for \( s > 0 \) in the main list in interval \( I_4 \)

<table>
<thead>
<tr>
<th>( q_3 )</th>
<th>( q_2 )</th>
<th>( r_4 )</th>
<th>( r_1 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coefficient</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 3\gamma_3/2 + (9\beta_2^{(4)} - 2\gamma_2)^2/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>5.39</td>
<td>a</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 8\gamma_3/3 + (10\beta_2^{(4)} - 2\gamma_2)^2/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.41</td>
<td>b</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 6\gamma_3/3 + 812\beta_2^{(4)} - 2\gamma_2)^2/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>3.55</td>
<td>c</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 11\gamma_3/4 + (13\beta_2^{(4)} - 2\gamma_2)^2/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>3.01</td>
<td>d</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 7\gamma_3/3 + (11\beta_2^{(4)} - 2\gamma_2)^2/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>3.45</td>
<td>e</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 12\gamma_3/4 + (12\beta_2^{(4)} - 2\gamma_2)^2/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>3.00</td>
<td>f</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 10\gamma_3/4 + (14\beta_2^{(4)} - 2\gamma_2)^2/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>3.04</td>
<td>g</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 15\gamma_3/5 + (15\beta_2^{(4)} - 2\gamma_2)^2/5 + \gamma_1/5 \leq h + \delta )</td>
<td>s=0</td>
<td>2.83</td>
<td>h</td>
</tr>
</tbody>
</table>

We now go through all cases from a) to h).

a) \( K(5) \) is represented by \( r_1 \) and \( s = 0 \). The last line of the corresponding inequality system reads

\[ \epsilon_4 + 3\gamma_3 + 5\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} \leq h + \delta. \]

From the system A) we choose the following lines:

\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5] \gamma_1 \leq h + \delta \]

Using the weight distribution 1, 2, 1 and 1 we get

\[ \epsilon_4 + 14\gamma_3/5 + 2\gamma_2/5 + 2\gamma_1/5 \leq h + \delta \]

yielding a coefficient 1.97 < 2.008 according to (41) and (43).
Alternatively we choose the following lines from system B):

\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5]\gamma_1 - \beta_1^{(4)} \leq h + \delta
\]

Using the weight distribution 1, 2, 1 and 1 we again get

\[
\epsilon_4 + 14\gamma_3/5 + 2\gamma_2/5 + 2\gamma_1/5 \leq h + \delta
\]

yielding a coefficient 1.97 < 2.008 according to (41) and (43).

b) \(K(5)\) is represented by \(r_1, r_4\) and \(s = 0\). If \(r_1\) comes first the last line of the list for \(K(5)\) reads

\[
\epsilon_4 + \gamma_3 + 5\beta_2^{(4)} - \gamma_2 + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4]\gamma_1 \leq h + \delta.
\]

From the system A) we choose the following line:

\[
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5]\gamma_1 \leq h + \delta.
\]

Averaging gives

\[
\epsilon_4 + 7\gamma_3/2 + 2\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta
\]

yielding a coefficient 1.89 < 2.008 according to (41) and (43).

If \(r_4\) comes first the first line of the list for \(K(5)\) reads

\[
\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4]\gamma_1 \leq h + \delta.
\]

From the system B) we choose the following line:

\[
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5]\gamma_1 - \beta_1^{(4)} \leq h + \delta.
\]

Averaging gives the same inequality and the same coefficient as above.

c) \(K(5)\) is represented by \(r_1, q_2\) and \(s = 0\). If \(r_1\) comes first the list for \(K(5)\) reads:

\[
\epsilon_4 + 2\gamma_3 + 4\beta_2^{(4)} - \gamma_2 + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + 5\beta_2^{(4)} - \gamma_2 + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta
\]

From the system A) we choose the following lines:

\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5]\gamma_1 \leq h + \delta
\]
Using the weight distribution 0, 1, 1, 3, 2 and 2 we get
\[ \epsilon_4 + 27\gamma_3/9 + 4\gamma_2/9 + 3\gamma_1/9 \leq h + \delta \]
yielding a coefficient 2.0019 < 2.008 according to (71).
If \( r_1 \) comes last the list for \( K(5) \) reads:
\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + 4\beta_2^{(4)} - \gamma_2 + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 5\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} & \leq h + \delta
\end{align*}
\]
From the system B) we choose the following lines:
\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + \beta_1^{(3)} - [1, 5] \gamma_1 - \beta_1^{(4)} & \leq h + \delta
\end{align*}
\]
Using the weight distribution 1, 1, 0, 3, 2 and 2 we again get the same inequality and the same coefficient as above.
d) \( K(5) \) is represented by \( r_1, r_4, q_2 \) and \( s = 0 \). If \( r_1 \) comes first we get still two alternatives for the list for \( K(5) \). We now assume that \( r_4 \) comes just before \( q_2 \). The corresponding line then reads:
\[ \epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + 2\beta_1^{(4)} - \gamma_1 \leq h + \delta. \]
From the system A) we choose the following line:
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta. \]
Avaraging using the weight distribution 1 and 2 gives
\[ \epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta. \]
yielding a coefficient 1.70 < 2.008 according to (41) and (43).
We now assume that \( q_2 \) comes just before \( r_4 \). The corresponding line then reads:
\[ \epsilon_4 + 5\gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta. \]
From the system A) we choose the following line:
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 4\beta_1^{(4)} - [0, 2] \gamma_1 \leq h + \delta. \]
Here we used that \( \kappa_1 \leq 0 \). Avaraging gives
\[ \epsilon_4 + 9\gamma_3/2 + (\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta. \]
yielding a coefficient $1.67 < 2.008$ according to (41) and (43).

If $r_1$ comes last we get still two alternatives for the list for $K(5)$. We now assume that $r_4$ comes just before $q_2$. The corresponding line then reads:

$$
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta.
$$

From the system B) we choose the following line:

$$
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} + [0, 2]\gamma_1 \leq h + \delta.
$$

Here we used that $\kappa_l \leq 0$. Avaraging gives

$$
\epsilon_4 + 7\gamma_3/2 + 7\gamma_2/2 + 7\gamma_1/2 \leq h + \delta.
$$

yielding a coefficient $1.14 < 2.008$ according to (41) and (43).

We now assume that $q_2$ comes just before $r_4$. The corresponding line then reads:

$$
\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4]\gamma_1 \leq h + \delta.
$$

From the system B) we choose the following line:

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
$$

Avaraging using the weight distribution 1 and 2 again gives

$$
\epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
$$

yielding a coefficient $1.70 < 2.008$ according to (41) and (43).

e) $K(5)$ is represented by $r_1, q_3$ and $s = 0$. If $r_1$ comes first the list we look at system A). Because of the main list we see that $[0, 2] = 1$. Therefore we have $2\beta_1^{(4)} \geq \gamma_1$ and thus we have $5\beta_1^{(4)} + \beta_1^{(3)} \geq 3\beta_1^{(4)} + \beta_1^{(3)} + \gamma_1$ and $[1, 5] \geq [1, 3] + 1$. We get two possibilities:$[1, 5] - [1, 3] = 2$ and $[1, 5] - [1, 3] = 1$.

In the first case the list for $K(5)$ reads:

$$
\epsilon_4 + 2\gamma_3 + 4\beta_2^{(4)} - \gamma_2 + \gamma_1 - \beta_1^{(4)} \leq h + \delta
$$

$$
\epsilon_4 + 4\gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + \gamma_3 + 5\beta_2^{(4)} - \gamma_2 + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 \leq h + \delta
$$

From the system A) we choose the following lines:

$$
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} + 2\beta_1^{(4)} - \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - \beta_1^{(3)} + [1, 5]\gamma_1 \leq h + \delta
$$
Using the weight distribution 0, 0, 2, 1 and 2 we get
\[\epsilon_4 + 15\gamma_3/5 + 7\beta^{(4)}_2/5 + (2[1, 5] - 2[1, 3] - 1)\gamma_1/5 = \epsilon_4 + 15\gamma_3/5 + 7\beta^{(4)}_2/5 + 3\gamma_1/5 \leq h + \delta\]
yielding a coefficient \(1.54 < 2.008\) according to (41) and (43).
If \([1, 5] - [1, 3] = 1\) we have to choose the following inequalities from B).
\[\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta^{(4)}_2 + \gamma_1 - \beta^{(4)}_1 \leq h + \delta \quad \epsilon_4 + 3\gamma_3 + 2\beta^{(4)}_2 + 5\beta^{(4)}_1 + \beta^{(3)}_1 - [1, 5]\gamma_1 \leq h + \delta\]

Using the weight distribution 0, 1, 0, 3 and 1 we get
\[\epsilon_4 + 14\gamma_3/5 + (3\gamma_2 - 2\beta^{(4)}_2)/5 + (3 + [1, 3] - [1, 5])\gamma_1/5 = \epsilon_4 + 14\gamma_3/5 + (3\gamma_2 - 2\beta^{(4)}_2)/5 + 2\gamma_1/5 \leq h + \delta\]
yielding a coefficient \(1.84 < 2.008\) according to (41) and (43).
If \(r_1\) comes last the list for \(K(5)\) reads:
\[\epsilon_4 + 4\gamma_3 + 2\beta^{(4)}_2 + \gamma_1 - 3\beta^{(4)}_1 - 3\beta^{(3)}_1 + [1, 3]\gamma_1 \leq h + \delta \quad \epsilon_4 + 2\gamma_3 + 4\beta^{(4)}_2 - \gamma_2 + 3\beta^{(4)}_1 + \beta^{(3)}_1 - [1, 3]\gamma_1 - \beta^{(4)}_1 \leq h + \delta \quad \epsilon_4 + 3\gamma_3 + 5\beta^{(4)}_2 - \gamma_2 + \beta^{(4)}_1 \leq h + \delta\]

We look at system B). Because of the main list we see that \([0, 2] = 0\). Therefore we have \(2\beta^{(4)}_1 < \gamma_1\) and thus we have \(5\beta^{(4)}_1 + \beta^{(3)}_1 < 3\beta^{(4)}_1 + \beta^{(3)}_1 + \gamma_1\) and \([1, 5] \leq [1, 3] + 1\). We get two possibilities: \([1, 5] - [1, 3] = 0\) and \([1, 5] - [1, 3] = 1\).
In the first case we choose the following lines from the system B)
\[\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta^{(4)}_2 + \gamma_1 - 2\beta^{(4)}_1 \leq h + \delta \quad \epsilon_4 + 2\gamma_3 + \beta^{(4)}_2 + 5\beta^{(4)}_1 + \beta^{(3)}_1 - [1, 5]\gamma_1 - \beta^{(4)}_1 \leq h + \delta\]

Using the weight distribution 2, 0, 0, 1 and 2 we get
\[\epsilon_4 + 15\gamma_3/5 + (\gamma_2 + 4\beta^{(4)}_2)/5 + (3 + 2[1, 3] - 2[1, 5])\gamma_1/5 = \epsilon_4 + 15\gamma_3/5 + (\gamma_2 + 4\beta^{(4)}_2)/5 + 3\gamma_1/5 \leq h + \delta\]
yielding a coefficient \(1.38 < 2.008\) according to (41) and (43).
If \([1, 5] - [1, 3] = 1\) we have to choose the following inequalities from B).
\[\epsilon_4 + 6\gamma_3 + \gamma_2 - 3\beta^{(4)}_2 + \gamma_1 - 5\beta^{(4)}_1 - \beta^{(3)}_1 + [1, 5]\gamma_1 \leq h + \delta \quad \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta^{(4)}_2 + \beta^{(4)}_1 \leq h + \delta\]
Using the weight distribution 0, 1, 0, 1 and 3 we again get
\[
\epsilon_4 + 14\gamma_3/5 + (3\gamma_2 - 2\beta_2^{(4)})/5 + (1 + [1, 5] - [1, 3])\gamma_1/5 = \\
\epsilon_4 + 14\gamma_3/5 + (3\gamma_2 - 2\beta_2^{(4)})/5 + 2\gamma_1/5 \leq h + \delta
\]
yielding a coefficient 1.84 < 2.008 according to (41) and (43).

f) \(K(5)\) is represented by \(r_1, r_4, q_3\) and \(s = 0\). If \(r_1\) comes first we get still two alternatives for the list for \(K(5)\). We now assume that \(r_4\) comes just before \(q_3\). The corresponding line then reads:
\[
\epsilon_4 + 4\gamma_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta.
\]
From the system A) we choose the following line:
\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta.
\]
Averaging gives
\[
\epsilon_4 + 7\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta.
\]
yielding a coefficient 1.14 < 2.008 according to (41) and (43).

We now assume that \(q_3\) comes just before \(r_4\). The corresponding line then reads:
\[
\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4]\gamma_1 \leq h + \delta.
\]
If this line substitutes the following line from the main list in A)
\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} + [0, 2]\gamma_1 \leq h + \delta
\]
we get there
\[
\epsilon_4 + 11\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)} + \beta_2^{(4)} - (\gamma_2 - 2\beta_2^{(4)}))/3 + \gamma_1/3 = \\
\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
\]
yielding a coefficient 1.89 < 2.008 according to (41) and (43).

If \(r_1\) comes last we get still two alternatives for the list for \(K(5)\). We now assume that \(q_3\) comes just before \(r_4\). The corresponding line then reads:
\[
\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4]\gamma_1 \leq h + \delta.
\]
From the system B) we choose the following line:
\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\]
Averaging gives
\[
\epsilon_4 + 7\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta.
\]
yielding a coefficient 1.14 < 2.008 according to (41) and (43).

We now assume that $r_4$ comes just before $q_3$. The corresponding line then reads:

$$
\epsilon_4 + 4\gamma_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta.
$$

If this line substitutes the following line from the main list in B)

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta
$$

we get there

$$
\epsilon_4 + 11\gamma_3 + (3\gamma_2 - 6\beta_2^{(4)} + 2\beta_2^{(4)} - (\gamma_2 - 2\beta_2^{(4)}))/3 + \gamma_1/3 = \\
\epsilon_4 + 11\gamma_3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
$$

yielding a coefficient 1.89 < 2.008 according to (41) and (43).

g) $K(5)$ is represented by $r_1, q_2, q_3$ and $s = 0$. If $r_1$ comes first we get still two alternatives for the list for $K(5)$. We now assume that $q_2$ comes just before $q_3$. The corresponding line then reads:

$$
\epsilon_4 + 4\gamma_3 + 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta.
$$

If this line substitutes the following line from the main list A)

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} + [0, 2]\gamma_1 \leq h + \delta
$$

we get there

$$
\epsilon_4 + 10\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)} + 2\beta_2^{(4)} - (\gamma_2 - 2\beta_2^{(4)}))/3 + \gamma_1/3 = \\
\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
$$

tyielding a coefficient 1.80 < 2.008 according to (41) and (43).

We now assume that $q_3$ comes just before $q_2$. The corresponding line then reads:

$$
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta.
$$

From the system A) we choose the following line:

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta.
$$

Avaraging gives

$$
\epsilon_4 + 6\gamma_3/2 + (\gamma_2 + \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.
$$

tyielding a coefficient 1.06 < 2.008 according to (41) and (43).
Now if \( r_1 \) comes last we have to look at system B). We still get two alternatives for the list for \( K(5) \). We now assume that \( q_2 \) comes just before \( q_3 \). The corresponding line then reads:

\[
\epsilon_4 + 4\gamma_3 + 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3]\gamma_1 \leq h + \delta.
\]

From the system B) (main list) we choose the following line:

\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\]

Avaraging gives

\[
\epsilon_4 + 6\gamma_3/2 + (\gamma_2 + \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.
\]

yielding a coefficient \( 1.06 < 2.008 \) according to (41) and (43).

We now assume that \( q_3 \) comes just before \( q_2 \). The corresponding line then reads:

\[
\epsilon_4 + 3\gamma_3 + 3\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1,3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2]\gamma_1 \leq h + \delta.
\]

If this line substitutes the following line from the main list B)

\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta
\]

we get there

\[
\epsilon_4 + 10\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)} + 3\beta_2^{(4)} - (\gamma_2 - \beta_2^{(4)}))/3 + \gamma_1/3 = \\
\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
\]

yielding a coefficient \( 1.80 < 2.008 \) according to (41) and (43).

h) \( K(5) \) is represented by \( r_1, q_2, q_3, r_4 \) and \( s = 0 \). Here the arguments from case f) and g) can be used to exclude this case. Either \( q_2 \) and \( q_3 \) are neighbours or \( q_3 \) and \( r_4 \) are neighbours.

**12.6 Interval \( I_5 \)**

In this interval we start considering the key number

\[
M(2) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (2\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1
\]

and try to find a representation for \( M(2) \) and all the numbers in the inverall \( K(2) = [M(2) - \gamma_1 + 1, M(2)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(1) \). Also here it turns out that only very few transfers are possible. No transfers from \( D^* \) or \( E \) can give optimal representaions as long as \( s > 0 \) since these transfers produce second terms larger than \( \gamma_2 \).
In interval $I_5$ we have $\frac{1}{3}\gamma_2 < \beta_2^{(4)} < \frac{2}{5}\gamma_2$. Therefore we have
$\gamma_2 > 5\beta_2^{(4)} - \gamma_2 > 2\beta_2^{(4)} > 4\beta_2^{(4)} - \gamma_2 > \beta_2^{(4)} > 6\beta_2^{(4)} - 2\gamma_2 > 3\beta_2^{(4)} - \gamma_2 > 0$ and only $r_1, r_3, r_4$ and $r_6$ from $A$ may be used. If $r_6$ is used we get
\[ \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 4\beta_2^{(4)} \leq h + \delta. \]

This contradicts (45) for $t = 6$ since $2\gamma_2 - 4\beta_2^{(4)} > \gamma_2/6$. Thus $s = 6$ is also impossible and we are left with $r_1, r_3, r_4$ and $s = 0, 1, 3$ and $s = 4$. Here $s = 3$ may be excluded since $2\beta_2^{(4)} - (3\beta_2^{(4)} - \gamma_2) > 3\beta_2^{(4)} - \gamma_2$. That means $s = 3$ could be used twice if it would end the list. If $r_3$ is used we see that $s = 1$ is impossible since $2\beta_2^{(4)} - (3\beta_2^{(4)} - \gamma_2) > \beta_2^{(4)}$.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \leq 4$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$. If $s_4 > 4$ we get
\[ \epsilon_4 + 5\gamma_3 + 2\beta_2^{(4)} \leq h + \delta. \]

This contradicts (45) for $t = 4$ since $2\beta_2^{(4)} > \gamma_2/2$.

### Table 21. List for $K(2)$ when $s > 0$ in interval $I_5$

<table>
<thead>
<tr>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=0</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=1</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=4</td>
<td>0.86</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 3\gamma_3/2 + 3\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>2.66</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=1</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.69</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 6\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.33</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=4</td>
<td>0.86</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 8\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3/2 + (\gamma_2 + \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.69</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.08</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 7\gamma_3/3 + (\gamma_2 + 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.19</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.75</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 7\gamma_3/3 + (\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.58</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.08</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.75</td>
</tr>
</tbody>
</table>
This gives us four cases.

a) $K(2)$ is represented by $r_1$ and $s = 0$.

b) $K(2)$ is represented by $r_1, r_4$ and $s = 0$.

c) $K(2)$ is represented by $r_1, r_3$ and $s = 0$.

d) $K(2)$ is represented by $r_1, r_3, r_4$ and $s = 0$.

All cases have $s = 0$. Thus we have $s = 0$ also for all later lists.

The arguments go into different directions. In case a) we continue with the main list whilst we continue with the list for $M(1)$ in the three other cases.

a) First we assume that $K(2)$ is represented by $r_1$ and $s = 0$ and look at the main list. The main list then can only contain $r_1, r_2, r_5$ and $s = 2$ or $s = 5$.

Below we go through all these cases.

<table>
<thead>
<tr>
<th>$r_5$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=2</td>
<td>1.44</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 6\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=5</td>
<td>1.12</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.47</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=2</td>
<td>1.44</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.87</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/3 + (4\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=5</td>
<td><strong>2.14</strong></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + (3\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.87</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 6\gamma_3/2 + 2\gamma_2 - 5\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=5</td>
<td>1.12</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (4\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=2</td>
<td><strong>2.14</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.47</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + (3\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=2</td>
<td>1.87</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (3\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=5</td>
<td>1.08</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (4\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=2</td>
<td><strong>2.14</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/3 + (4\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=5</td>
<td><strong>2.14</strong></td>
</tr>
</tbody>
</table>

In all cases where the coefficient is above 2.008 we get the same inequality

$$\epsilon_4 + 11\gamma_3/3 + (4\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.$$  

We combine this inequality with the inequality from the representation of $K(2)$

$$\epsilon_4 + 3\gamma_3/2 + 3\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$$

using the weights 9 and 16 and get

$$\epsilon_4 + 57\gamma_3/25 + 12\gamma_2/25 + 11\gamma_1/25 \leq h + \delta.$$
yielding a coefficient 1.98 < 2.008 according to (41) and (43). Thus case a) is finished.

b) Here $K(2)$ is represented by $r_1, r_4$ and $s = 0$. Two orderings are possible. We call them A) and B).

A)
\[
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4] \gamma_1 \leq h + \delta.
\]

and

B)
\[
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4] \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\]

Now we consider the key number $M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1$

and try to find a representation for $M(1)$ and all the numbers in the inverall $K(1) = [M(1) - \gamma_1 + 1, M(1)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(1)$. Also here it turns out that only very few transfers are possible. No transfers from $D^*$ or $E$ can give optimal representations as long as $s > 0$ since these transfers produce second terms larger than $\gamma_2$.

We already know that $r_3$ and $r_6$ from $A$ may be used.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 = 1$ the second term will be $\gamma_2$ which is impossible since $s > 0$. If $s_4 = 3$ we get
\[
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} \leq h + \delta.
\]

This contradicts (45) for $t = 3$ since $2\gamma_2 - 2\beta_2^{(4)} > 4\gamma_2/5$. If $s_4 = 4$ we get
\[
\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} \leq h + \delta.
\]

This contradicts (45) for $t = 4$ since $2\gamma_2 - 3\beta_2^{(4)} > \gamma_2/2$. And finally, if $s_4 \geq 5$ we get
\[
\epsilon_4 + 6\gamma_3 + \beta_2^{(4)} \leq h + \delta.
\]

This contradicts (45) for $t = 5$ since $\beta_2^{(4)} > \gamma_2/3$.

Thus we are left with $r_3, r_6, q_2$ and $s = 0$. 

Thus we have five choices for \( K(1) \):

I) \( K(1) \) is represented by \( s = 0 \) alone.

II) \( K(1) \) is represented by \( r_3 \) and \( s = 0 \).

III) \( K(1) \) is represented by \( r_3, r_6 \) and \( s = 0 \).

IV) \( K(1) \) is represented by \( r_3, q_2 \) and \( s = 0 \).

V) \( K(1) \) is represented by \( r_3, r_6, q_2 \) and \( s = 0 \).

I) In this case the main list has length 1 with a constant term \( \gamma_1 - 1 \). This means that the transfer \((0,1,0)\) does not give a positive gain. In this cases we have

\[
\gamma_3 + \gamma_2 + \gamma_1 \leq h + \delta.
\]

Combining this inequality with \( \epsilon_4 + 8\gamma_3/3 + (\beta_2^{(4)} + \gamma_2)/3 + \gamma_1/3 \leq h + \delta \) for \( K(2) \) and \( \epsilon_4 + \gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta \) for \( K(1) \) using the weights 2, 5 and 5 yields

\[
\epsilon_4 + 61\gamma_3/30 + 53\beta_2^{(4)}/90 + 26\gamma_1/30 \leq 1.2h + \delta
\]

giving a coefficient \( 1.99 < 2.008 \) according to (42).

II) Here \( K(1) \) is represented by \( r_3 \) and \( s = 0 \). We get the following list:

\[
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta
\]

\[
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 \leq h + \delta.
\]

Combing inequality system A) with this system using the weight distribution 0, 1, 0 and 0, 1 yields

\[
\epsilon_4 + 6\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]

giving a coefficient \( 1.97 < 2.008 \) according to (41) and (43).

Combing inequality system B) with this system using the weight distribution 0, 1, 0 and 1, 0 yields the same inequality and the same coefficient as above.

### Table 23. List for \( K(1) \) when \( s > 0 \) in the main list in interval \( I_5 \)

<table>
<thead>
<tr>
<th>( q_2 )</th>
<th>( r_6 )</th>
<th>( r_3 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + \gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>3.0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 5\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>2.36</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 8\gamma_3/2 + (2\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>1.77</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 12\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>2.16</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>2.0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 8\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>2.41</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 11\gamma_3/3 + (3\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>1.93</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 15\gamma_3/4 + (4\gamma_2 - 7\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>( s = 0 )</td>
<td>2.24</td>
</tr>
</tbody>
</table>
III) Here $K(1)$ is represented by $r_3, r_6$ and $s = 0$. We get two orderings. The first list reads

\[
\begin{align*}
\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 6]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 + 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weight distribution 1, 2 and 0 yields

\[
\epsilon_4 + 15\gamma_3/3 + (4\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]
giving a coefficient 1.91 < 2.008 according to (41) and (43).

Using the other ordering the list reads

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 6]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weight distribution 0, 2 and 1 yields the same inequality and coefficient as above. Thus we finished case III).

IV) Here $K(1)$ is represented by $q_2, r_3$ and $s = 0$. We get two orderings. The first list reads

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Combing inequality system A) with this system using the weight distribution 1, 0, 0 and 1, 0, 0 yields

\[
\epsilon_4 + 5\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient 1.6 < 2.008 according to (41) and (43).

Combing inequality system B) with this system using the weight distribution 0, 1, 0 and 1, 0, 0 yields

\[
\epsilon_4 + 6\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.
\]
giving a coefficient 1.97 < 2.008 according to (41) and (43).

Now we consider the other ordering for the list for $K(1)$.

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [2, 1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [2, 1]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [3, 1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [3, 1]\gamma_1 & \leq h + \delta.
\end{align*}
\]
Combing inequality system A) with this system using the weight distribution 0, 1, 1 and 0, 1, 0 yields
\[ \epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.93 \( < \) 2.008 according to (41) and (43).

Combing inequality system B) with this system using the weight distribution 0, 0, 1 and 0, 1, 0 yields
\[ \epsilon_4 + 5\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta. \]
giving a coefficient 1.6 \( < \) 2.008 according to (41) and (43) and we finished IV).

V) Here \( K(1) \) is represented by \( q_2, r_3, r_6 \) and \( s = 0 \). We get six orderings. The first ordering is:
\[ \alpha ) \ r_6, r_3, q_2 \text{ and } s = 0. \]
Here the first two lines of the inequality system read
\[ \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta \]
Using the weights 1 and 2 yields
\[ \epsilon_4 + 15\gamma_3/3 + (4\gamma_2 - 9\beta_1^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.91 \( < \) 2.008 according to (41) and (43).
\[ \beta ) q_2, r_3, r_6 \text{ and } s = 0. \]
Here the last two lines of the inequality system read
\[ \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 \leq h + \delta \]
Using the weights 2 and 1 again yields
\[ \epsilon_4 + 15\gamma_3/3 + (4\gamma_2 - 9\beta_1^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving the same coefficient as above.
\[ \gamma ) r_3, r_6, q_2 \text{ and } s = 0. \]
Here the list reads
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta \]
Using the weights 0, 2, 1 and 1 yields
\[ \epsilon_4 + 18\gamma_3/4 + (5\gamma_2 - 10\beta_4^{(4)})/4 + \gamma_1/4 \leq h + \delta \]
giving a coefficient 2.0 < 2.008 according to (41) and (43).
\( \delta \) \( q_2, r_6, r_3 \) and \( s = 0 \).
Here the list reads
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 \leq h + \delta \]

Using the weights 1, 1, 2 and 0 yields
\[ \epsilon_4 + 18\gamma_3/4 + (5\gamma_2 - 10\beta_4^{(4)})/4 + \gamma_1/4 \leq h + \delta \]
giving a coefficient 2.0 < 2.008 according to (41) and (43).
\( \delta \) \( r_6, q_2, r_3 \) and \( s = 0 \).
Here the list reads
\[ \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [2, 6]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 \leq h + \delta \]

Here we have to involve the inequality systems A) and B).
Combing inequality system A) with this system using the weight distribution 0, 1, 0 and 0, 0, 0, 1 yields
\[ \epsilon_4 + 6\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.97 < 2.008 according to (41) and (43).
Combing inequality system B) with this system using the weight distribution 0, 0, 1 and 0, 0, 1, 0 yields
\[ \epsilon_4 + 5\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq 1h + \delta \]
giving a coefficient 1.6 < 2.008 according to (41) and (43).
\( \epsilon \) \( r_3, q_2, r_6 \) and \( s = 0 \).
Here the list reads
\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} &- \beta_1^{(3)} + [1,3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} &+ \beta_1^{(3)} - [1,3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} &+ \beta_1^{(3)} - [1,2]\gamma_1 - 6\beta_1^{(4)} - 2\beta_1^{(3)} + [2,6]\gamma_1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} &+ 2\beta_1^{(3)} - [2,6]\gamma_1 \leq h + \delta
\end{align*}
\]

Here again we have to involve the inequality systems A) and B).
Combing inequality system A) with this system using the weight distribution 0, 0, 1 and 0, 0, 1, 0 yields
\[
\epsilon_4 + 8\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq 1h + \delta.
\]
giving a coefficient $1.0 < 2.008$ according to (41) and (43).
Combing inequality system B) with this system using the weight distribution 0, 1, 0 and 1, 0, 0, 0 yields
\[
\epsilon_4 + 6\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.
\]
giving a coefficient $1.97 < 2.008$ according to (41) and (43). Thus we finished case b).
c) We now consider the case where $K(2)$ is represented by $r_1, r_3$ and $s = 0$. Here $K(4)$ has the same representation and $K(1)$ gets an additional contribution. The amount is $\beta_2^{(4)}$. We now look at $K(1)$. We already found that there are five possibilities for $K(1)$. But we know that the list for $K(1)$ contains $r_3$. Therefore we only have to look at the last four possibilities.
II) $K(1)$ is represented by $r_3$ and $s = 0$. Since we get an additional contribution we have
\[
\begin{align*}
\epsilon_4 + 5\gamma_3/2 + (\gamma_2 - \beta_2^{(4)} + \beta_2^{(4)})/2 + \gamma_1/2 &= \\
\epsilon_4 + 5\gamma_3/2 + \gamma_2/2 + \gamma_1/2 &\leq h + \delta
\end{align*}
\]
giving a coefficient $1.6 < 2.008$ according to (41) and (43)).
III) $K(1)$ is represented by $r_3, r_6$ and $s = 0$. Since we get an additional contribution we have
\[
\begin{align*}
\epsilon_4 + 12\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)} + \beta_2^{(4)})/3 + \gamma_1/3 &= \\
\epsilon_4 + 12\gamma_3/3 + (3\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 &\leq h + \delta
\end{align*}
\]
giving a coefficient $1.77 < 2.008$ according to (41) and (43)).
IV) $K(1)$ is represented by $r_3, q_2$ and $s = 0$. Since we get an additional contribution we have
\[
\begin{align*}
\epsilon_4 + 8\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)} + \beta_2^{(4)})/3 + \gamma_1/3 &= \\
\epsilon_4 + 8\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 &\leq h + \delta
\end{align*}
\]
This is not enough to exclude the case so we take the inequality system for $K(2)$ into account. The first ordering is the following:

$$
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3] \gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1,3] \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
$$

This implies that the inequality system for $K(1)$ has to look like this:

$$
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3] \gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1,3] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2] \gamma - 1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2] \gamma_1 \leq h + \delta.
$$

Using the weight distribution 1, 1, 0 and 0, 1, 0 yields

$$
\epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$
giving a coefficient $1.76 < 2.008$ according to (41) and (43).

The other ordering is the following:

$$
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3] \gamma_1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1,3] \gamma_1 \leq h + \delta.
$$

This implies that the inequality system for $K(1)$ has to look like this:

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2] \gamma_1 \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2] \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3] \gamma - 1 \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1,3] \gamma_1 \leq h + \delta.
$$

Using the weight distribution 0, 1, 1 and 0, 1, 0 again yields

$$
\epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$
giving a coefficient $1.76 < 2.008$ according to (41) and (43).

V) $K(1)$ is represented by $q_2, r_3, r_6$ and $s = 0$. Since we get an additional contribution we have

$$
\epsilon_4 + 15\gamma_3/4 + (4\gamma_2 - 7\beta_2^{(4)} + \beta_2^{(4)})/4 + \gamma_1/4 = \epsilon_4 + 15\gamma_3/4 + (4\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta.
$$
giving a coefficient $1.96 < 2.008$ according to (41) and (43) and case c) is finished.
d) We now consider the case where $K(2)$ is represented by $r_1, r_3, r_4$ and $s = 0$. We go through all the six possible orderings of the list

$\alpha) K(2)$ is represented by $r_1, r_4, r_3$ and $s = 0$

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2(4) + \gamma_1 - \beta_1(4) & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2(4) + \beta_4(4) - 4\beta_1(4) - \beta_1(3) + [1, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2(4) + 4\beta_1(4) + \beta_3(3) - [1, 4]\gamma_1 - 3\beta_1(4) - \beta_1(3) + [1, 3]\gamma - 1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2(4) + 3\beta_4(4) + \beta_3(3) - [1, 3]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weight distribution $1, 0, 1$ and $0$ we get

\[
\epsilon_4 + 6\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]

giving a coefficient $1.33 < 2.008$ according to (41) and (43).

$\beta) K(2)$ is represented by $r_3, r_4, r_1$ and $s = 0$

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2(4) + \gamma_1 - 3\beta_1(4) - \beta_1(3) + [1, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2(4) + 3\beta_1(4) + \beta_3(3) - [1, 3]\gamma_1 - 4\beta_1(4) - \beta_1(3) + [1, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2(4) + 4\beta_1(4) + \beta_3(3) - [1, 4]\gamma_1 - \beta_1(4) & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2(4) + \beta_1(4) & \leq h + \delta.
\end{align*}
\]

Using the weight distribution $0, 1, 0$ and $1$ we get

\[
\epsilon_4 + 6\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]

giving a coefficient $1.33 < 2.008$ according to (41) and (43).

$\gamma) K(2)$ is represented by $r_1, r_3, r_4$ and $s = 0$

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2(4) + \gamma_1 - \beta_1(4) & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2(4) + \beta_4(4) - 3\beta_1(4) - \beta_1(3) + [1, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2(4) + \beta_1(4) + [1, 3]\gamma_1 - 4\beta_1(4) - \beta_1(3) + [1, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2(4) + 4\beta_1(4) + \beta_3(3) - [1, 4]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weight distribution $0, 1, 2$ and $1$ we get

\[
\epsilon_4 + 15\gamma_3/4 + (3\gamma_2 - 3\beta_2(4))/4 + \gamma_1/4 \leq h + \delta
\]

giving a coefficient $1.84 < 2.008$ according to (41) and (43).

$\delta) K(2)$ is represented by $r_4, r_3, r_1$ and $s = 0$

\[
\begin{align*}
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2(4) + \gamma_1 - 4\beta_1(4) - \beta_1(3) + [1, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2(4) + 4\beta_1(4) + \beta_3(3) - [1, 4]\gamma_1 - 3\beta_1(4) - \beta_1(3) + [1, 3]\gamma - 1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2(4) + 3\beta_1(4) + \beta_3(3) - [1, 3]\gamma_1 - \beta_1(4) & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2(4) + \beta_1(4) & \leq h + \delta.
\end{align*}
\]
Using the weight distribution 1, 2, 1 and 0 we get

$$
\epsilon_4 + 15\gamma_3/4 + (3\gamma_2 - 3\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
$$
giving a coefficient 1.84 < 2.008 according to (41) and (43).

c) $K(2)$ is represented by $r_4, r_1, r_3$ and $s = 0$

\begin{align*}
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1,4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1,4]\gamma - 1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + [1,3]\gamma - 1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1,3]\gamma_1 & \leq h + \delta.
\end{align*}

This is not enough to exclude the case so we take the inequality system for $K(1)$ into account. We know there are five possibilities for $K(1)$. But since $r_3$ occurs in the list for $K(2)$ it also has to occur in the list for $K(1)$ and the first possibility for $K(1)$ cannot occur. The choices II) and III) can be excluded in a similar way as above. Since the list for $K(2)$ contains $r_4, r_1, r_3$ and $s = 0$, we know that the list for $K(4)$ contains at least 3 lines, giving an additional contribution to the list for $K(1)$. The amount of the contribution is $3\beta_2^{(4)} - \gamma_2$.

If we choose II) for $K(1)$ the inequality

$$
\epsilon_4 + 5\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
$$

turns into

$$
\epsilon_4 + 5\gamma_3/2 + 2\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta
$$
giving a coefficient 2.007 < 2.008 according to (71). If we choose III) for $K(1)$ the inequality

$$
\epsilon_4 + 12\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$

turns into

$$
\epsilon_4 + 12\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$
giving a coefficient 1.96 < 2.008 according to (41) and (43). Now we look at choice IV for $K(1)$. Because of the chosen ordering of the list for $K(2)$ the ordering of the transfers for list for $K(1)$ is also given, namely $q_2, r_3$ and $s = 0$.

The second line in this list now reads:

$$
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta.
$$

Combining this line with the above inequality system for $K(2)$ using the weights 0, 0, 1, 1 and 1 yields

$$
\epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$
We now combine the above list for \( K \) in the same way as above, since we know we get an additional contribution to the first possibility for \( K_0 \) and \( K_1 \). We get

\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6] \gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6] \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3] \gamma - 1 \leq h + \delta \]

\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3] \gamma_1 \leq h + \delta. \]

Using the weights 1, 1, 2 and 0 yields

\[ \epsilon_4 + 18\gamma_3/4 + (5\gamma_2 - 10\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \]

giving a coefficient 2.0 < 2.008 according to (41) and (43).

The last ordering for \( K(2) \) in case d) is

\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3] \gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3] \gamma_1 - \beta_1^{(4)} \leq h + \delta \]

\[ \epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta \]

\[ \epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4] \gamma_1 \leq h + \delta. \]

Again this is not enough to exclude the case so we take the inequality system for \( K(1) \) into account. We know there are five possibilities for \( K(1) \). But since \( r_3 \) occurs in the list for \( K(2) \) it also has to occur in the list for \( K(1) \) and the first possibility for \( K(1) \) cannot occur. The choices II) and III) can be excluded in the same way as above, since we know we get an additional contribution to the list for \( K(1) \) from the one for \( K(4) \). In case IV) we know that \( r_3 \) stands in the first line in the list for \( K(1) \). Line two therefore reads

\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta. \]

We now combine the above list for \( K(2) \) with this line using the weights 1, 1, 0, 0 and 1. We get

\[ \epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]

giving a coefficient 1.76 < 2.008 according to (41) and (43).

We now look at the last alternative V) for \( K(1) \). Since \( r_3 \) occurs in the first line in the list for \( K(2) \) it must have the same position in the list for \( K(1) \) and we
only get two possible orderings there. If we choose the ordering \( r_3, q_2, r_6 \) and \( s = 0 \) for \( K(1) \) then again \( q_2 \) follows after \( r_3 \) and we get the same line as in the argument above. If we choose the last ordering \( r_3, r_6, q_2 \) and \( s = 0 \) for \( K(1) \) we get

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma - 1 & \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 6\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 6]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 6\beta_1^{(4)} + 2\beta_1^{(3)} + [2, 6]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} - [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights \( 0, 2, 1 \) and \( 1 \) yields

\[
\epsilon_4 + 18\gamma_3/4 + (5\gamma_2 - 10\beta_2^{(4)})/4 + \gamma_1/4 \leq 1h + \delta
\]

giving a coefficient \( 2.0 < 2.008 \) according to \((41)\) and \((43)\). Thus we finished case \( d \).

### 12.7 Interval \( I_6 \)

In this interval we start considering the key number

\[
M(2) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (2\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1
\]

and try to find a representation for \( M(2) \) and all the numbers in the inverall \( K(2) = [M(2) - \gamma_1 + 1, M(2)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(2) \). Also here it turns out that only very few transfers are possible. No transfers from \( D^* \) or \( E \) can give optimal representaions as long as \( s > 0 \) since these transfers produce second terms larger than \( \gamma_2 \).

In interval \( I_6 \) we have \( \frac{5}{3}\gamma_2 < \beta_2^{(4)} < \frac{1}{2}\gamma_2 \). Therefore we have \( \gamma_2 > 2\beta_2^{(4)} > 4\beta_2^{(4)} - \gamma_2 > 6\beta_2^{(4)} - 2\gamma_2 > \beta_2^{(4)} > 3\beta_2^{(4)} - \gamma_2 > 5\beta_2^{(4)} - 2\gamma_2 > 0 \) and only \( r_1, r_3, r_4, r_5 \) and \( r_6 \) from \( A \) may be used. If \( r_5 = 1 \) we get

\[
\epsilon_4 + 6\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} \leq h + \delta.
\]

This contradicts \((45)\) for \( t = 5 \) since \( 2\gamma_2 - 3\beta_2^{(4)} > \gamma_2/3 \). Thus \( s = 5 \) is also impossible and we are left with \( r_1, r_3, r_4, r_6 \) and \( s = 0, 1, 3, 4 \) and \( s = 6 \). Here \( s = 3 \) may be excluded since \( 2\beta_2^{(4)} - (3\beta_2^{(4)} - \gamma_2) > 3\beta_2^{(4)} - \gamma_2 \). That means \( s = 3 \) could be used twice if it would end the list. If \( r_3 \) is used we see that \( s = 1 \) is impossible since \( 2\beta_2^{(4)} - (3\beta_2^{(4)} - \gamma_2) > \beta_2^{(4)} \).

Now we assume that \( (s_2, s_3, s_4) \in B^* \) is used. For all \( 1 \leq s_4 \leq 6 \) the second term will be \( \geq \gamma_2 \) which is impossible since \( s > 0 \).
Table 24 A. List for $K(2)$ when $s > 0$ in the main list in interval $I_6$,
Part I $r_6 = 0$

<table>
<thead>
<tr>
<th>$r_6$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>s</th>
<th>Coeff.</th>
<th>case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=0</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=1</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=4</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=6</td>
<td>0.96</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 3\gamma_3/2 + 3\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>2.22</td>
<td>a</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=1</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.85</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3/2 + 3\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.33</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.85</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=4</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.65</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 8\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.18</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.85</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.85</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 14\gamma_3/3 + (3\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.95</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 5\gamma_3/2 + (\gamma_2 + \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.14</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/2 + (3\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 7\gamma_3/3 + (\gamma_2 + 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.06</td>
<td>c</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.93</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 13\gamma_3/3 + (3\gamma_2 - 4\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.64</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/3 + (4\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.51</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 12\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>2.16</td>
<td>d</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.89</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 18\gamma_3/4 + (4\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=6</td>
<td>2.00</td>
<td></td>
</tr>
</tbody>
</table>
Thus we have six choices for $K(2)$. In all of them we have $s = 0$. Therefore we must have $s = 2$ in the main list. In the first three cases a), b) and c) we proceed by regarding the main list.

In the last three cases we proceed by regarding $K(4)$ and $K(1)$.

Since the first three cases only contain $r_1, r_2, r_3$ and $r_4$ and $s = 0$ we know that the corresponding main lists may contain $r_1, r_2, r_3$ and $r_4$ and will end with $s = 2$.  

**Table 24 B. List for $K(2)$ when $s > 0$ in the main list in interval $I_6$, Part II $r_6 = 1$**

<table>
<thead>
<tr>
<th>$r_6$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 8\gamma_3/2 + (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.65</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + 2\gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=6</td>
<td>0.96</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.64</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 14\gamma_3/3 + (3\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>2.03</td>
<td>e</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 9\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 13\gamma_3/3 + (3\gamma_2 - 4\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.64</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 14\gamma_3/3 + (3\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=1</td>
<td>1.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.95</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 12\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.38</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/3 + (4\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.70</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/2 + (3\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=6</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 14\gamma_3/4 + (3\gamma_2 - 2\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>1.85</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 18\gamma_3/4 + (4\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 13\gamma_3/3 + (3\gamma_2 - 4\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.64</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 17\gamma_3/4 + (4\gamma_2 - 5\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>1.79</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/3 + (4\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.70</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/3 + (4\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/3 \leq h + \delta$</td>
<td>s=6</td>
<td>1.70</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 19\gamma_3/5 + (4\gamma_2 - 4\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta$</td>
<td>s=0</td>
<td>2.10</td>
<td>f</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 18\gamma_3/4 + (4\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 18\gamma_3/4 + (4\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=6</td>
<td>2.00</td>
<td></td>
</tr>
</tbody>
</table>
The corresponding main list cannot contain both \( r_3 \) and \( r_4 \). Otherwise both transfers would occur in the list for \( K(2) \). Here we go through these possibilities for the main list.

**Table 25. Main list for case a), b) and c) when \( s > 0 \) in interval \( I_6 \)**

<table>
<thead>
<tr>
<th>( r_4 )</th>
<th>( r_3 )</th>
<th>( r_2 )</th>
<th>( r_1 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>1.89*</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>2.59</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>1.89*</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>2.59</td>
<td>I</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 9\gamma_3/3 + (4\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>1.92 *</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>2.59</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 9\gamma_3/3 + (4\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>1.92*</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>2.47</td>
<td>II</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 10\gamma_3/3 + (4\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>2.73</td>
<td>III</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>2.47</td>
<td>IV</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 10\gamma_3/3 + (4\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>( s=2 )</td>
<td>2.73</td>
<td>V</td>
</tr>
</tbody>
</table>

a) \( M(2) \) is represented by \( r_1 \) and \( s = 2 \). Then the main list may contain \( r_1, r_2 \) and at the end of the main list we have \( s = 2 \) and only case I) can occur.

I) If the main list consists of \( r_1, r_2 \) and \( s = 2 \) we combine the average inequality

\[
\epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]

from the main list with

\[
\epsilon_4 + 3\gamma_3/2 + 3\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta
\]

from the list for \( K(2) \) and get

\[
\epsilon_4 + 8\gamma_3/4 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]

giving a coefficient \( 2.0 < 2.008 \) according to (41) and (43).

Thus we finished case a).

b) \( K(2) \) is represented by \( r_1, r_4 \) and \( s = 0 \). Then the main list may contain

\( r_1, r_4 \) and possibly \( r_2 \) and at the end of the main list we have \( s = 2 \). All cases I) to V) apply here.
I) $K(0)$ is represented by $r_1, r_2$ and $s = 2$. The line in the middle of the main list reads
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta^{(4)}_2 + \beta^{(4)}_1 \leq h + \delta. \]
If $r_1$ occurs in the first line of the $K(2)$-list the first line there reads
\[ \epsilon_4 + 2\gamma_3 + \beta^{(4)}_2 + \gamma_1 - \beta^{(4)}_1 \leq h + \delta. \]
Combining the two lines yields
\[ \epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient $2.0 < 2.008$ according to (41) and (43). If $r_4$ stands first in the $K(2)$-list the last lines in the two lists read: Main list:
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta^{(4)}_2 + [0, 3] \gamma_1 - \beta^{(4)}_1 \leq h + \delta. \]
$K(2)$-list:
\[ \epsilon_4 + \gamma_3 + 2\beta^{(4)}_2 + \beta^{(4)}_1 \leq h + \delta. \]
Combining the two lines yields
\[ \epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient $2.0 < 2.008$ according to (41) and (43) and we finished case I).
II) $K(0)$ is represented by $r_4$ and $s = 2$. The average inequality reads
\[ \epsilon_4 + 8\gamma_3/2 + (3\beta^{(4)}_2 - 6\beta^{(4)}_2) + 2 + \gamma_1/2 \leq h + \delta. \]
Combining this inequality with the average inequality for the $K(2)$-list
\[ \epsilon_4 + 8\gamma_3/3 + (\gamma_2 + \beta^{(4)}_2) + 2 + \gamma_1/2 \leq h + \delta \]
using the weights 2 and 3 gives the following
\[ \epsilon_4 + 16\gamma_3/5 + (4\gamma_2 - 5\beta^{(4)}_2) + 2 + \gamma_1/5 \leq h + \delta \]
yielding a coefficient $1.96 < 2.008$ according to (71) and we finished case II).
III) $K(0)$ is represented by $r_1, r_4$ and $s = 2$. Here we have $2\beta^{(4)}_1 = \gamma_1$. If $r_1$ comes first in the main list the first line there reads
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta^{(4)}_2 + \gamma_1/2 \leq h + \delta. \]
giving a coefficient $2.0 < 2.008$ according to (41) and (43). If $r_4$ comes first in the main list then the last line in the main list reads
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta^{(4)}_2 + \beta^{(4)}_1 - 2\beta^{(4)}_1 + [0, 3] \gamma_1 \leq h + \delta. \]
The last line in the \( K(2) \)-list reads

\[
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\]

Combining these lines again yields

\[
\epsilon_4 + 4\gamma/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient \( 2.0 < 2.008 \) according to (41) and (43) and we finished case III).

IV) \( K(0) \) is represented by \( r_2, r_4 \) and \( s = 2 \). Here we get the same average inequalities as in case II and can use the same argument as there.

V) \( K(0) \) is represented by \( r_2, r_1, r_4 \) and \( s = 2 \). If \( r_1 \) comes before \( r_4 \) in the main list the second line there reads

\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta
\]

whilst the first line in the \( K(2) \)-list reads

\[
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta.
\]

Combining these lines again yields

\[
\epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient \( 2.0 < 2.008 \) according to (41) and (43).

If \( r_4 \) comes before \( r_1 \) in the main list then the last line in the main list reads

\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} + \gamma_1 \leq h + \delta.
\]

The last line in the \( K(2) \)-list reads

\[
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\]

Combining these lines again yields

\[
\epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient \( 2.0 < 2.008 \) according to (41) and (43) and we finished case V).

Thus we finished case b).

c) \( M(2) \) is represented by \( r_1, r_3 \) and \( s = 0 \). Then the main list must contain \( r_1, r_2 \) and at the end of the main list we have \( s = 2 \). The other main lists containing \( r_3 \) have all coefficients below \( 2.008 \). So we look at case I). The line in the middle of the main list reads

\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\]
If $r_1$ occurs in the first line of the $K(2)$-list the first line there reads

$$\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta.$$  

Combining the two lines yields

$$\epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta$$

giving a coefficient $2.0 < 2.008$ according to (41) and (43). If $r_3$ stands first in the $K(2)$-list the last lines in the two lists read: Main list

$$\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + [0,3][\gamma_1 - \beta_1^{(4)} \leq h + \delta.$$  

$K(2)$-list

$$\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.$$  

Combining the two lines yields

$$\epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta$$

giving a coefficient $2.0 < 2.008$ according to (41) and (43) and we finished c). For the next three cases d), e) and f) we include $K(4)$ in the argument. Therefore we have to look at the key number

$$M(4) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (4\beta_2^{(4)} - \gamma_2 - 1)a_2 + \gamma_1 - 1$$

and try to find a representation for $M(4)$ and all the numbers in the interval

$$K(4) = [M(4) - \gamma_1 + 1, M(4)].$$

First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(1)$. Also here it turns out that only very few transfers are possible. No transfers from $D^*$ or $E$ can give optimal representations as long as $s > 0$ since these transfers produce second terms larger than $\gamma_2$.

Only $r_1, r_3, r_5$ and $r_6$ from $A$ may be used. If $r_5 = 1$ or $s = 5$ we get

$$\epsilon_4 + 6\gamma_3 + \gamma_2 - \beta_2^{(4)} \leq h + \delta.$$  

This contradicts (45) for $t = 5$ since $\gamma_2 - \beta_2^{(4)} > \gamma_2/3$. Since $s = 0$ for $K(2)$ we also have $s = 0$ for $K(4)$.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. The second term will be $\gamma_2$ which is impossible since $s > 0$. The only exception is $q_2$. Thus we are left with $r_1, r_3, r_6, q_2$ and $s = 0$. We know $r_1$ is used. We cannot have $q_2$ and $r_6$ in the same list otherwise we get

$$\epsilon_4 + 3\gamma_3 + 2\beta_2^{(4)} \leq h + \delta$$

and

$$\epsilon_4 + 7\gamma_3 + \gamma_2 - 2\beta_2^{(4)} \leq h + \delta.$$
Together these inequalities give
\[ \epsilon_4 + 5\gamma_3 + \gamma_2/2 \leq h + \delta. \]
contradicting (45) for \( t = 4 \).

Table 26. List for \( K(4) \) when \( s > 0 \) in the main list in interval \( I_6 \), \( r_1 = 1 \)

<table>
<thead>
<tr>
<th>( q )</th>
<th>( r_6 )</th>
<th>( r_3 )</th>
<th>( r_1 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 3\gamma_3/2 + (7\beta_2^{(4)} - 2\gamma_2)/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>3.25</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 7\gamma_3/3 + (8\beta_2^{(4)} - 2\gamma_2)/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.75</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 10\gamma_3/3 + (5\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.13</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 14\gamma_3/4 + (6\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>2.25</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 6\gamma_3/3 + (9\beta_2^{(4)} - 2\gamma_2)/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.64</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 10\gamma_3/4 + (10\beta_2^{(4)} - 2\gamma_2)/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>2.59</td>
<td>6</td>
</tr>
</tbody>
</table>

d) Here \( K(2) \) is represented by \( r_1, r_3, r_4 \) and \( s = 0 \). Since \( K(4) \) has to contain \( r_1 \) and \( r_3 \) only case 2), 4) and 6) apply. Case 4) and 6) can be excluded easily. But case 2) has to be regarded for every ordering of \( K(2) \). In case 4) \( K(4) \) is represented by \( r_1, r_3, r_6 \) and \( s = 0 \). Because of the additional contribution from \( K(2) \) the average inequality for \( K(4) \)

\[ \epsilon_4 + 14\gamma_3/4 + (6\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 \leq h + \delta. \]

turns into

\[ \epsilon_4 + 14\gamma_3/4 + (2\beta_2^{(4)} + \gamma_2)/4 + \gamma_1/4 \leq h + \delta. \]
giving a coefficient \( 1.97 < 2.008 \) according to (41) and (43).

In case 6) we even have to include \( K(1) \). We therefore have to look at the key number

\[ M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1 \]

and try to find a representation for \( M(1) \) and all the numbers in the interval \( K(1) = [M(1) - \gamma_1 + 1, M(1)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(1) \). Also here it turns out that only very few transfers are possible. No transfers from \( D^* \) or \( E \) can give optimal representations as long as \( s > 0 \) since these transfers produce second terms larger than \( \gamma_2 \).

Only \( r_3 \) and \( r_5 \) from \( A \) may be used. Since \( s = 0 \) for \( K(2) \) we also have \( s = 0 \) for \( K(1) \).

Now we assume that \( (s_2, s_3, s_4) \in B^* \) is used. For \( s_4 = 1 \) and \( s_4 = 3 \) the second term will be \( \geq \gamma_2 \) which is impossible since \( s > 0 \). For \( s_4 = 4 \) we get

\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} \leq h + \delta \]
contradicting (45) for \( t = 4 \) since \( 2\gamma_2 - 3\beta_2^{(4)} \geq \gamma_2/2 \). For \( s_4 = 4 \) and \( s_4 = 6 \) we get

\[ \epsilon_4 + 6\gamma_3 + \beta_2^{(4)} \leq h + \delta \]

contradicting (45) for \( t = 5 \) since \( 2\gamma_2 - 3\beta_2^{(4)} \geq \gamma_2/3 \).

Thus we are left with \( r_3, r_5, q_2 \) and \( s = 0 \). We know that \( r_3 \) is used.

**Table 27. List for \( K(1) \) when \( s > 0 \) in the main list in interval \( I_6 \), \( r_3 = 1 \)**

<table>
<thead>
<tr>
<th>( q_2 )</th>
<th>( r_5 )</th>
<th>( r_3 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 5\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>2.59</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 11\gamma_3/3 + (3\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.48</td>
<td>( \beta )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 8\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.66</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 14\gamma_3/4 + (4\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>2.57</td>
<td>( \delta )</td>
</tr>
</tbody>
</table>

In case 6) only \( \gamma \) and \( \delta \) apply and we get an additional contribution from the \( K(4) \)-list to the \( K(1) \)-list. The amount of the contribution is

\[
2(3\beta_2^{(4)} - \gamma_2) = 6\beta_2^{(4)} - 2\gamma_2.
\]

In case \( \gamma \) the average inequality

\[
\epsilon_4 + 8\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]

turns into

\[
\epsilon_4 + 8\gamma_3/3 + 4\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta
\]

giving a coefficient \( 1.98 < 2.008 \) according to (41) and (43).

In case \( \delta \) the average inequality

\[
\epsilon_4 + 14\gamma_3/4 + (4\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
\]

turns into

\[
\epsilon_4 + 14\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta
\]

giving a coefficient \( 1.85 < 2.008 \) according to (41) and (43).

We now go through the different orderings of the inequality system for \( K(2) \). In the cases where we need information from \( K(4) \) we know that only case 2) is possible.

A) \( K(2) \) is represented by \( r_1, r_4, r_3 \) and \( s = 0 \)

\[
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta
\]
\[
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(3)} - \beta_1^{(3)} + [1, 4]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 \leq h + \delta.
\]
Using the weights 1, 0, 1 and 0 yields

\[ \epsilon_4 + 6\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq 1h + \delta \]

giving a coefficient 1.32 < 2.008 according to (41) and (43).

B) \(K(2)\) is represented by \(r_3, r_4, r_1\) and \(s = 0\)

\[
\begin{align*}
&\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3] \gamma_1 \leq h + \delta \\
&\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3] \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta \\
&\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4] \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
&\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\end{align*}
\]

Using the weights 0, 1, 0 and 1 yields

\[ \epsilon_4 + 6\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq 1h + \delta \]

giving a coefficient 1.32 < 2.008 according to (41) and (43).

C) \(K(2)\) is represented by \(r_1, r_3, r_4\) and \(s = 0\)

\[
\begin{align*}
&\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
&\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3] \gamma_1 \leq h + \delta \\
&\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3] \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta \\
&\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4] \gamma_1 \leq h + \delta.
\end{align*}
\]

Here we have to include \(K(4)\) in the argument. We already know that only case 2) applies.

In case 2) \(K(4)\) is represented by \(r_1, r_3\) and \(s = 0\). Since \(r_1\) is at the top of the list for \(K(2)\) it will have the same position in the list for \(K(4)\). Thus the last line in the list for \(K(4)\) reads

\[ \epsilon_4 + \gamma_3 + 4\beta_2^{(4)} - \gamma_2 + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3] \gamma_1 \leq h + \delta. \]

Combining the list for \(K(2)\) with this line using the weights 0, 1, 0 and 1 gives

\[ \epsilon_4 + 10\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta. \]

giving a coefficient 1.74 < 2.008 according to (41) and (43).

D) \(K(2)\) is represented by \(r_4, r_3, r_1\) and \(s = 0\)

\[
\begin{align*}
&\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4] \gamma_1 \leq h + \delta \\
&\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4] \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3] \gamma_1 \leq h + \delta \\
&\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3] \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
&\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.
\end{align*}
\]
Here again we have to include \( K(4) \) in the argument. Again we know that only case 2) applies.

In case 2) \( K(4) \) is represented by \( r_1, r_3 \) and \( s = 0 \). Since \( r_1 \) is in the second last line in the list for \( K(2) \) it will have the same position in the list for \( K(4) \). Thus the first line in the list for \( K(4) \) reads

\[
\epsilon_4 + 4\gamma_3 + \beta_2(4) + \gamma_1 - 3\beta_1(4) - \beta_1(3) + [1,3]\gamma_1 \leq h + \delta.
\]

Combining the list for \( K(2) \) with this line using the weights 0, 1, 1, 0 and 1 gives

\[
\epsilon_4 + 10\gamma_3/3 + (\gamma_2 + \beta_2(4))/3 + \gamma_1/3 \leq h + \delta.
\]

giving a coefficient 1.74 < 2.008 according to (41) and (43).

E) \( K(2) \) is represented by \( r_1, r_2, r_3 \) and \( s = 0 \)

\[
\epsilon_4 + 5\gamma_3/2 + (\gamma_2 - \beta_2(4))/2 + \gamma_1/2 \leq h + \delta
\]

because of the additional contribution from \( K(4) \). This gives a coefficient 1.95 < 2.008 according to (41) and (43).

\[
\epsilon_4 + 11\gamma_3/3 + (3\gamma_2 - 5\beta_2(4))/3 + \gamma_1/3 \leq h + \delta
\]

because of the additional contribution from \( K(4) \). This gives a coefficient 1.93 < 2.008 according to (41) and (43).

\[
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2(4) + \gamma_1 - \beta_1(4) \leq h + \delta
\]
Combining the list for $K(2)$ with this line using the weights 0, 0, 1, 1 and 1 gives
\[\epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.\]
giving a coefficient $1.85 < 2.008$ according to (41) and (43).

\(\delta\) Here the transfer $r_3$ is in the second last line of the $K(4)$-list. Therefore it has to be in the same position for the $K(1)$-list, too. But since $r_5$ enters the list we have two orderings. If $q_2$ comes just before $r_3$ we can argue in the same way as above.

If the ordering is $q_2, r_5, r_3$ and $s = 0$ we get the following inequality system for $K(1)$.

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 2\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 5\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 5]\gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Averaging using the weights 1, 0, 1 and 0 yields
\[\epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.\]
giving a coefficient $1.85 < 2.008$ according to (41) and (43).

F) $K(2)$ is represented by $r_3, r_1, r_4$ and $s = 0$

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} + 4\beta_1^{(4)} - \beta_1^{(3)} + [1, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + \beta_1^{(3)} - [1, 4]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Here again we have to include $K(4)$ in the argument. Again we know that only case 2) applies.

In case 2) $K(4)$ is represented by $r_1, r_3$ and $s = 0$. In addition we have to study $K(1)$.

Case $\alpha$) and $\beta$) are excluded as in case E).

\(\gamma\) Here the transfer $r_3$ is in the first line of the $K(4)$-list. Therefore it has to be in the same position for the $K(1)$-list, too. Line 2 in this list then reads
\[\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta.\]

Combining the list for $K(2)$ with this line using the weights 1, 1, 0, 0 and 1 gives
\[\epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.\]
giving a coefficient 1.85 < 2.008 according to (41) and (43).

\[ \epsilon_4 + 4\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - \beta_1^{(3)} + [1, 3]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 6\gamma_3 + 2\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1, 3]\gamma_1 - 5\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta. \]

Averaging using the weights 0, 1, 0 and 1 yields

\[ \epsilon_4 + 7\gamma_3/2 + (2\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta. \]

giving a coefficient 1.85 < 2.008 according to (41) and (43).

e) Here \( K(2) \) is represented by \( r_1, r_4, r_6 \) and \( s = 0 \). When we look at \( K(4) \) only the cases 3) and 4) apply since they contain \( r_6 \).

In case 3 the average inequality for \( K(4) \) is

\[ \epsilon_4 + 10\gamma_3/3 + (5\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 \leq h + \delta. \]

Now \( K(2) \) has length 4 and therefore we get an additional contribution to \( K(4) \), the amount of which is \( 2(\gamma_2 - 2\beta_2^{(4)}) \). Thus the inequality turns into

\[ \epsilon_4 + 10\gamma_3/3 + (\beta_2^{(4)} + \gamma_2)/3 + \gamma_1/3 \leq h + \delta. \]

giving a coefficient 1.74 < 2.008 according to (41) and (43).

In case 4 the average inequality for \( K(4) \) is

\[ \epsilon_4 + 14\gamma_3/4 + (6\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 \leq h + \delta. \]

Now \( K(2) \) has length 4 and therefore we get an additional contribution to \( K(4) \), the amount of which is \( 2(\gamma_2 - 2\beta_2^{(4)}) \). Thus the inequality turns into

\[ \epsilon_4 + 14\gamma_3/4 + (2\beta_2^{(4)} + \gamma_2)/4 + \gamma_1/4 \leq h + \delta. \]

giving a coefficient 1.97 < 2.008 according to (41) and (43). Thus we finished case e).

f) Here \( K(2) \) is represented by \( r_1, r_3, r_4, r_6 \) and \( s = 0 \). Again we include \( K(4) \) in the argument. From the above list for \( K(4) \) we see that the only possibility is that \( K(4) \) is represented by \( r_6, r_3, r_1 \) and \( s = 0 \) having the average inequality

\[ \epsilon_4 + 14\gamma_3/4 + (6\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 \leq h + \delta. \]
Now $K(2)$ has length 5 and therefore there is an additional contribution to $K(4)$, the amount of which is $3(\gamma_2 - 2\beta_2^{(4)})$. Thus the inequality turns into

$$\epsilon_4 + 14\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta.$$  

giving a coefficient $1.85 < 2.008$ according to (41) and (43). Thus we finished interval $I_6$.

### 12.8 Interval $I_7$

In this interval it suffices to consider one single key number

$$M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1$$

and try to find a representation for $M(1)$ and all the numbers in the interval $K(1) = [M(1) - \gamma_1 + 1, M(1)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(1)$. Also here it turns out that only very few transfers are possible. No transferes from $D^*$ or $E$ can give optimal representaions as long as $s > 0$ since these transferes produce second terms $\geq \gamma_2$.

In interval $I_7$ we have $\frac{1}{2}\gamma_2 < \beta_2^{(4)} < \frac{3}{5}\gamma_2$. Therefore we have $\gamma_2 > 5\beta_2^{(4)} - 2\gamma_2 > 3\beta_2^{(4)} - \gamma_2 > \beta_2^{(4)} > 6\beta_2^{(4)} - 3\gamma_2 > 4\beta_2^{(4)} - \gamma_2 > 2\beta_2^{(4)} - \gamma_2 > 0$ and only $r_2, r_4$ and $r_6$ from $A$ may be used. Below we go through all these possibilities. We know that $s = 1$ is impossible and therefore $s = 0, 2, 4$ or $s = 6$. Since $\beta_2^{(4)} - (2\beta_2^{(4)} - \gamma_2) > 2\beta_2^{(4)} - \gamma_2$ we see that $s = 2$ is impossible, since the corresponding transfer $(s_2, s_3, 2) \in C$ could be used twice at the end of the list. Only the second coefficient is reduced but after such a reduction there is still enough space to perform another one. So $s = 2$ cannot produce the optimal representation at the end of the list for $M(1)$.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 = 1$ or $s_4 = 2$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$. If $s_4 = 3$ we get

$$\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} \leq h + \delta.$$  

This contradicts (45) for $t = 3$ since $2\gamma_2 - 2\beta_2^{(4)} > 4\gamma_2/5$. For $s_4 > 3$ we have $\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} \leq h + \delta$, yielding a coefficient $1.98 < 2.008$ according to (41) and (43). Thus we are left with the possibilities $r_2, r_4, r_6$ and $s = 0, 4, 6$. All these possibilities are listed up in the table below. Only two cases have coefficients above 2.008.
Thus there are only two cases left:

If \( r_2 \) and \( r_4 \) are used we get two orderings.

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4] \gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 0, 2 and 1 yields \( \epsilon_4 + 11\gamma_3 + (4\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \)

giving a coefficient 1.89 < 2.008 according to (41) and (43).

The other ordering gives
\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta. \]

Using the weights 1, 2 and 0 yields again
\[ \epsilon_4 + 11\gamma_3/3 + (4\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \] and we are finished.

Now we turn to the case where \(r_2, r_4\) and \(r_6\) are used for \(K(1)\). Here we get 6 orderings.

A) If \(K(1)\) is represented by \(r_2, r_6, r_4\) and \(s = 0\) we get
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 7\gamma_3 + 3\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 \leq h + \delta. \]

Using the weights 0, 1, 0 and 1 yields \(\epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\)
giving a coefficient \(1.38 < 2.008\) according to (41) and (43).

B) If \(K(1)\) is represented by \(r_2, r_4, r_6\) and \(s = 0\) we get
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 7\gamma_3 + 3\gamma_2 - 5\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 \leq h + \delta. \]

Using the weights 0, 0, 3 and 1 yields
\[ \epsilon_4 + 22\gamma_3/4 + (9\gamma_2 - 14\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \] giving a coefficient \(1.86 < 2.008\) according to (41) and (43).

C) If \(K(1)\) is represented by \(r_6, r_2, r_4\) and \(s = 0\) we get
\[ \epsilon_4 + 7\gamma_3 + 3\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 \leq h + \delta. \]
Using the weights 0, 1, 2 and 0 yields
\[ \epsilon_4 + 13\gamma_3/3 + (5\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1, 75 < 2.008 according to (41) and (43).

D) If \( K(1) \) is represented by \( r_6, r_4, r_2 \) and \( s = 0 \) we get
\[
\begin{align*}
\epsilon_4 &+ 7\gamma_3 + 3\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 < h + \delta \\
\epsilon_4 &+ 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 < h + \delta \\
\epsilon_4 &+ 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 < h + \delta \\
\epsilon_4 &+ \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 < h + \delta.
\end{align*}
\]

Using the weights 1, 3, 0 and 0 yields
\[ \epsilon_4 + 22\gamma_3/4 + (9\gamma_2 - 14\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \]
giving a coefficient 1, 86 < 2.008 according to (41) and (43).

E) If \( K(1) \) is represented by \( r_4, r_2, r_6 \) and \( s = 0 \) we get
\[
\begin{align*}
\epsilon_4 &+ 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 < h + \delta \\
\epsilon_4 &+ 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 < h + \delta \\
\epsilon_4 &+ 7\gamma_3 + 3\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 < h + \delta \\
\epsilon_4 &+ \gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 < h + \delta.
\end{align*}
\]

Using the weights 1, 2, 0 and 0 yields
\[ \epsilon_4 + 11\gamma_3/3 + (4\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1, 89 < 2.008 according to (41) and (43).

F) If \( K(1) \) is represented by \( r_4, r_6, r_2 \) and \( s = 0 \) we get
\[
\begin{align*}
\epsilon_4 &+ 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 < h + \delta \\
\epsilon_4 &+ 7\gamma_3 + 3\gamma_2 - 5\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 < h + \delta \\
\epsilon_4 &+ 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 < h + \delta \\
\epsilon_4 &+ \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 < h + \delta.
\end{align*}
\]

Using the weights 0, 1, 0 and 1 yields
\[ \epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1, 38 < 2.008 according to (41) and (43).

Thus we are finished with Interval \( I_7 \).

### 12.9 Interval \( I_8 \)

We start considering the key number
\[ M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1 \]
and try to find a representation for $M(1)$ and all the numbers in the inverall
$K(1) = [M(1) - \gamma_1 + 1, M(1)]$. First we check which transfers from the sets
$A, B^*, C, D^*$ or $E$ can be used in the interval $K(1)$. Also here it turns out that
only very few transfers are possible. No transfers from $D^*$ and $E$ can be used
since $s > 0$.

In interval $I_8$ we have \( \frac{2}{3} \gamma_2 < \beta_2^{(4)} \). Therefore we have
\[
\gamma_2 > 3\beta_2^{(4)} - \gamma_2 > 6\beta_2^{(4)} - 3\gamma_2 > \beta_2^{(4)} > 4\beta_2^{(4)} - 2\gamma_2 > 2\beta_2^{(4)} - \gamma_2 > 5\beta_2^{(4)} - 3\gamma_2 > 0
\]
and only $r_2, r_4$ and $r_5$ from $A$ may be used. If $r_5$ is used we have
\[
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} \leq h + \delta.
\]
This contradicts (45) for $t = 5$ since $3\gamma_2 - 4\beta_2^{(4)} > \gamma_2/3$.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \leq 2$ the second term will be
\[
\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} \leq h + \delta.
\]
This contradicts (45) for $t = 4$ since $\beta_2^{(4)} > \gamma_2/2$. Thus only $s_4 = 3$ may be
possible.

and we are left with the possibilities $r_2, r_4, q_3$ and $s = 0, 2, 4$. Of course $s = 1$ is
impossible. But also $s = 2$ has to be excluded since $\beta_2^{(4)} - 2(2\beta_2^{(4)} - \gamma_2) > 0$.

This means that $s = 2$ cannot produce the minimal representation at the end of
the list for $K(1)$.

If $q_3$ is used we cannot have $s = 4$ since $\beta_2^{(4)} - (3\beta_2^{(4)} - 2\gamma_2) > (4\beta_2^{(4)} - 2\gamma_2)$.

This means that $q_3$ would not have given the minimal representation.

| Table 29. $K(1)$ for $s > 0$ in interval $I_8$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $q_3$ | $r_4$ | $r_2$ | Average inequality | $s$ | Coefficient |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0 | 0 | 0 | $\epsilon_4 + \gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta$ | $s=0$ | 1.66 |
| | | | $\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta$ | $s=4$ | 1.35 |
| 0 | 0 | 1 | $\epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta$ | $s=0$ | 2.0 |
| | | | $\epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$ | $s=4$ | 1.88 |
| 0 | 1 | 0 | $\epsilon_4 + 6\gamma_3/2 + (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$ | $s=0$ | 1.85 |
| | | | $\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta$ | $s=4$ | 1.35 |
| 0 | 1 | 1 | $\epsilon_4 + 9\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$ | $s=0$ | **2.37** |
| | | | $\epsilon_4 + 8\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$ | $s=4$ | 1.88 |
| 1 | 0 | 0 | $\epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$ | $s=0$ | 1.2 |
| 1 | 0 | 1 | $\epsilon_4 + 8\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$ | $s=0$ | 1.92 |
| 1 | 1 | 0 | $\epsilon_4 + 10\gamma_3/3 + (4\gamma_2 - 4\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$ | $s=0$ | 1.80 |
| 1 | 1 | 1 | $\epsilon_4 + 13\gamma_3/4 + (5\gamma_2 - 5\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$ | $s=0$ | **2.22** |
There are only two cases left for $K(1)$: $r_2, r_4$ and $r_2, r_4, q_3$. In both cases the regular representation stands at the end of the list. This has also to be the case for later lists.

We now continue with $K(3)$. If it is possible to get one addtional contribution from $K(3)$ to $K(1)$ then we are "saved". The first inequality then reads

$$\epsilon_4 + 9\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)} + 2\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 = \epsilon_4 + 9\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$$

giving a coefficient $2.001 < 2.008$ according to (41) and (43). The second inequality then reads

$$\epsilon_4 + 13\gamma_3/4 + (5\gamma_2 - 5\beta_2^{(4)} + 2\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 = \epsilon_4 + 13\gamma_3/4 + (4\gamma_2 - 3\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$$

giving a coefficient $1.99 < 2.008$ according to (41) and (43).

We now try to find a representation for $M(3)$ and all the numbers in the interval $K(3) = [M(3) - \gamma_1 + 1, M(3)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(3)$. Also here it turns out that only very few transfers are possible. No transfers from $D^*$ and $E$ can be used since $s > 0$.

If $r_5$ is used we have

$$\epsilon_4 + 6\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} \leq h + \delta.$$

This contradicts (45) for $t = 5$ since $2\gamma_2 - 2\beta_2^{(4)} > \gamma_2/3$.

Thus only $r_1, r_2, r_4$ and $r_6$ from $A$ may be used.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \leq 3$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$. If $s_4 \geq 4$ we get

$$\epsilon_4 + 5\gamma_3 + 3\beta_2^{(4)} - \gamma_2 \leq h + \delta.$$

This contradicts (45) for $t = 4$ since $3\beta_2^{(4)} - \gamma_2 > \gamma_2/2$. Thus no transfers from $B^*$ are possible and we are left with the possibilities $r_1, r_2, r_4, r_6$ and $s = 0, 1, 4, 6$. Of course $s = 3$ is impossible. But also $s = 2$ has to be excluded since $3\beta_2^{(4)} - \gamma_2 - 2(2\beta_2^{(4)} - \gamma_2) > 0$ This means that $s = 2$ cannot produce the minimal representation at the end of the list for $K(3)$.

We now divide the overview over the possible cases into two parts, one with $r_6$ and one without.
Table 30 A. \( K(3) \) for \( s > 0 \) in interval \( I_8 \), Part I, \( r_6 = 0 \)

<table>
<thead>
<tr>
<th>( r_6 )</th>
<th>( r_4 )</th>
<th>( r_2 )</th>
<th>( r_1 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - \gamma_2 + \gamma_1 \leq h + \delta )</td>
<td>s=0</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 \leq h + \delta )</td>
<td>s=1</td>
<td>1.90*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=4</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 7\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=6</td>
<td>0.89</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 3\gamma_3/2 + (5\beta_2^{(4)} - 2\gamma_2)/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>2.66</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 \leq h + \delta )</td>
<td>s=1</td>
<td>1.90*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 7\gamma_3/2 + \beta_2^{(4)} + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 9\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=6</td>
<td>1.69</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 4\gamma_3/2 + (4\beta_2^{(4)} - \gamma_2)/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>1.42</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 5\gamma_3/2 + (3\beta_2^{(4)} - \gamma_2)/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=1</td>
<td>1.95</td>
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<tr>
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<td>( \epsilon_4 + 8\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 10\gamma_3/2 + (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=6</td>
<td>1.11</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 6\gamma_3/3 + (6\beta_2^{(4)} - 2\gamma_2)/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.64</td>
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<td>s=1</td>
<td>1.82</td>
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<td>( \epsilon_4 + 10\gamma_3/3 + 2\beta_2^{(4)} + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=6</td>
<td>1.77</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 6\gamma_3/2 + 2\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>1.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 7\gamma_3/2 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=1</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 5\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=4</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td>( \epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=6</td>
<td>1.25</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 8\gamma_3/3 + (4\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.18</td>
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<td>( \epsilon_4 + 7\gamma_3/2 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=1</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 7\gamma_3/2 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 14\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=6</td>
<td>1.79</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 9\gamma_3/3 + 3\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>1.60</td>
</tr>
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<td></td>
<td></td>
<td>( \epsilon_4 + 10\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=1</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 8\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 15\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=6</td>
<td>1.42</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 11\gamma_3/4 + (5\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>2.36</td>
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<td></td>
<td></td>
<td>( \epsilon_4 + 10\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=1</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 10\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 17\gamma_3/4 + (2\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=6</td>
<td>1.89</td>
</tr>
</tbody>
</table>

In the 3 last "surviving" cases the length of the lists is \( \geq 3 \). Therefore we get at least one additional contribution to the list \( K(1) \)-list and we are through. Thus we are left with \( r_1 \) and \( s = 0 \) for \( K(3) \). We now look at the the remaining cases.
with \( r_6 = 1 \).

**Table 30 B.** \( K(3) \) for \( s > 0 \) in interval \( I_8 \) including \( r_6 = 1 \)

<table>
<thead>
<tr>
<th>( r_6 )</th>
<th>( r_4 )</th>
<th>( r_2 )</th>
<th>( r_1 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 8\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 9\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=1</td>
<td>1.51</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 7\gamma_3/2 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=6</td>
<td>0.89</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 10\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 9\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=1</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 14\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.72</td>
</tr>
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<td></td>
<td>( \epsilon_4 + 9\gamma_3/2 + (\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=6</td>
<td>1.69</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 11\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>1.44</td>
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<td></td>
<td></td>
<td>( \epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=1</td>
<td>1.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 15\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.42</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 10\gamma_3/2 + (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=6</td>
<td>1.11</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 13\gamma_3/4 + 3\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>2.12</td>
</tr>
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<td>( \epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=1</td>
<td>1.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 17\gamma_3/4 + (2\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=4</td>
<td>1.89</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=6</td>
<td>1.77</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 13\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>1.39</td>
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<tr>
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<td></td>
<td></td>
<td>( \epsilon_4 + 14\gamma_3/3 + (2\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=1</td>
<td>1.79</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>( \epsilon_4 + 12\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=6</td>
<td>1.25</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 15\gamma_3/4 + (\gamma_2 + \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>1.96</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td>( \epsilon_4 + 14\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=1</td>
<td>1.79</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 14\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.79</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 14\gamma_3/3 + (2\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=6</td>
<td>1.79</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 16\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>1.62</td>
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<td>( \epsilon_4 + 17\gamma_3/4 + (2\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=1</td>
<td>1.89</td>
</tr>
<tr>
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<td></td>
<td>( \epsilon_4 + 15\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.42</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>( \epsilon_4 + 15\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=6</td>
<td>1.42</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 18\gamma_3/5 + (\gamma_2 + 2\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta )</td>
<td>s=0</td>
<td>2.11</td>
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<td></td>
<td>( \epsilon_4 + 17\gamma_3/4 + (2\gamma_2\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=1</td>
<td>1.89</td>
</tr>
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<td></td>
<td>( \epsilon_4 + 17\gamma_3/4 + (2\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=4</td>
<td>1.89</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 17\gamma_3/4 + (2\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=6</td>
<td>1.89</td>
</tr>
</tbody>
</table>

We get two cases for \( K(3) \): \( r_1, r_2, r_6 \) and \( s = 0 \) and \( r_1, r_2, r_4, r_6 \) and \( s = 0 \). In the second case either \( r_2 \) or \( r_4 \) comes first or last or they appear one after the
other. In any case this gives an additional contribution to the list for $K(1)$ and again we are through.

If $r_2$ comes first or last in the first case we again get an additional contribution to the list of $K(1)$ and we are through. So the first case can only survive if $r_2$ appears in the middle giving the orderings $r_1, r_2, r_6, s = 0$ and $r_6, r_2, r_1, s = 0$.

All together we now have two situations: One where $K(3)$ is represented by $r_1$ and $s = 0$ and one where $K(3)$ is represented by $r_6, r_2, r_1$ and $s = 0$. In the last case $r_2$ has to stand between $r_6$ and $r_1$.

We start with the very first case for $K(3)$ namely $r_1, s = 0$. We now look at the main list. If $r_5$ is used we have

$$\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} \leq h + \delta.$$ 

This contradicts (45) for $t = 5$ since $4\gamma_2 - 5\beta_2^{(4)} > \gamma_2/3$.

Thus only $r_1, r_2, r_3, r_4$ and $r_6$ from $A$ may be used. Now $r_2, r_4, r_6$ do not occur in the list for $K(3)$ so they can not be part of the main list either. Otherwise they would have to turn up in this list, too. So there are only few possibilities for the main list:

1) $s = 3$ giving
\[\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 \leq h + \delta,\]

giving a coefficient $1.68 < 2.008$ according to (41) and (43).

2) $r_3, s = 3$ with the same inequality and coefficient.

3) $r_1, s = 3$

4) $r_3, r_1, s = 3$

Case 3) and 4) are treated simultaneously.

In case 3) we use $(0, 0, 1)$ and $(\lfloor 1, 3 \rfloor, 1, 3)$ and get the following main list

\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta\]
\[\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + \lfloor 1, 3 \rfloor \gamma_1 \leq h + \delta\]

Now since $s = 3$ we have $\kappa_1 = \kappa_2 = 3\beta_1^{(4)} + \beta_1^{(3)} - \lfloor 1, 3 \rfloor \gamma_1 \leq 0$ or
\[2\beta_1^{(4)} + \beta_1^{(3)} - \lfloor 1, 3 \rfloor \gamma_1 \leq -\beta_2^{(4)}\]
and the system now reads

\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + (1 - \lfloor 1, 3 \rfloor) \gamma_1 + 2\beta_1^{(4)} + \beta_1^{(3)} \leq h + \delta\]
\[\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + \lfloor 1, 3 \rfloor \gamma_1 \leq h + \delta\]

(73)

In case 4) the main list reads

\[\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + \lfloor 1, 3 \rfloor \gamma_1 \leq h + \delta\]
\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \lfloor 1, 3 \rfloor \gamma_1 + 3\beta_1^{(4)} + \beta_1^{(3)} - \beta_1^{(4)} \leq h + \delta\]
\[\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - \beta_1^{(3)} + \lfloor 1, 3 \rfloor \gamma_1 \leq h + \delta\]
and we see that the last two lines coincide with (73). We now combine one of these lines with lines from the list for $K(1)$.

If $K(1)$ is represented by $r_2, r_4$ and $s = 0$ we have two orderings. We write $([1,2],1,2)$ for $r_2$ and $([2,4],2,4)$ for $r_4$.

$\alpha$) If $r_2$ is first we get

$$\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2]\gamma_1 \leq h + \delta.$$ 

Combining this with the first line in (73) we get

$$\epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$$

giving a coefficient $2.007 < 2.008$ according to (41) and (43).

$\beta$) If $r_2$ stands in the second last line we get for the very last line

$$\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2]\gamma_1 \leq h + \delta.$$ 

Combining this with the last line in (73) we get the same inequality and the same coefficient as above.

We now look at the second possibility for $K(1)$ namely $r_2, r_4, q_3$. We write $([2,3],2,3)$ when $q_3$ is used. Here we get six orderings, but we need not consider them all separately. If $r_2$ comes first the argument $\alpha$ will apply. If $r_2$ comes in the second last line argument $\beta$ will apply. Thus $r_2$ comes on place two. If $r_2$ comes just before $r_4$ we get

$$\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2,4]\gamma_1 \leq h + \delta$$

Combining this with the first line in (73) we get

$$\epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$$

giving a coefficient $1.89 < 2.008$ according to (71).

If $r_4$ comes just before $r_2$ we get

$$\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2,4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2]\gamma_1 \leq h + \delta$$

Combining this with the last line in (73) we get the same inequality and the same coefficient as above.

Now we consider the case where $M(3)$ is represented by $r_1, r_2, r_6$ and $s = 0$.

Here the list for $M(6)$ has at least length 3 and thus will give an additional contribution to $M(1)$. The amount is $6\beta_2^{(4)} - 3\gamma_2 - \beta_2^{(4)}$. The corresponding average inequality with the additional contribution then reads

$$\epsilon_4 + 13\gamma_3/4 + (5\gamma_2 - 5\beta_2^{(4)} + 5\beta_2^{(4)} - 3\gamma_2)/4 + \gamma_1/4 = \epsilon_4 + 13\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta$$

giving a coefficient $1.99 < 2.008$ according to (41) and (43).
Thus we are left with the case where $M(1)$ is represented by $r_2, r_4$ and $s = 0$ and $M(3)$ is represented by $r_1, r_2, r_6$ and $s = 0$. We now combine the corresponding lists and exclude all remaining cases. Since $r_2$ cannot appear first or last in the list for $M(3)$ - we already mentioned this - we just have two orderings for this list. We write $([3, 6], [3, 6])$ when $r_6$ is used.

$$
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \gamma_1 - \beta_1^{(4)} \leq h + \delta
$$
$$
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(3)} - \beta_1^{(4)} + [1, 2]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 + 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - \gamma_2 + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 \leq h + \delta
$$

Now there are two orderings for $K(1)$:

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(3)} - \beta_1^{(4)} + [1, 2]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 \leq h + \delta
$$

Using the weights 0, 0, 1, 0 and 0, 0, 1 yields

$$
\epsilon_4 + 8\gamma_3 + 2 + (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
$$

giving a coefficient 1.31 < 2.008 according to (41) and (43). The second ordering for $M(1)$ gives the following inequality system:

$$
\epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 4]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta
$$

Using the weights 0, 0, 1, 0 and 0, 1, 1 yields

$$
\epsilon_4 + 11\gamma_3 + 3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$

giving a coefficient 1.89 < 2.008 according to (41) and (43). The second ordering for $M(3)$ gives the following inequality system:

$$
\epsilon_4 + 7\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 6]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 6]\gamma_1 - 2\beta_1^{(3)} - \beta_1^{(4)} + [1, 2]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + 2\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - \gamma_2 + \beta_1^{(4)} \leq h + \delta
$$
Now again there are two orderings for $K(1)$:
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(3)} - \beta_1^{(4)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 \leq h + \delta \]

Using the weights 0, 1, 0, 0 and 1, 1, 0 yields
\[ \epsilon_4 + 11\gamma_3/3 + (3\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]

giving a coefficient $1.89 < 2.008$ according to (41) and (43). The second ordering for $M(1)$ gives the following inequality system:
\[ \epsilon_4 + 5\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 4]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - 1\beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta \]

Using the weights 0, 1, 0, 0 and 1, 0, 0 gives
\[ \epsilon_4 + 8\gamma_3/2 + (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]

giving a coefficient $1.31 < 2.008$ according to (41) and (43).
Thus we finished interval $I_8$.

### 12.10 Interval $I_9$

Again we start considering the key number
\[ M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1 \]

and try to find a representation for $M(1)$ and all the numbers in the interval $K(1) = [M(1) - \gamma_1 + 1, M(1)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(1)$. Also here it turns out that only very few transfers are possible. No transfers from $D^*$ and $E$ can be used since $s > 0$.

In interval $I_9$ we have $\frac{2}{3}\gamma_2 < \beta_2^{(4)} < \frac{3}{4}\gamma_2$. Therefore we have
\[ \gamma_2 > 4\beta_2^{(4)} - 2\gamma_2 > \beta_2^{(4)} > 5\beta_2^{(4)} - 3\gamma_2 > 2\beta_2^{(4)} - \gamma_2 > 6\beta_2^{(4)} - 4\gamma_2 > 3\beta_2^{(4)} - 2\gamma_2 > 0 \]

and only $r_2, r_3, r_5$ and $r_6$ from $A$ may be used. If $r_6$ is used we have
\[ \epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} \leq h + \delta. \]

This contradicts (45) for $t = 6$ since $4\gamma_2 - 5\beta_2^{(4)} > \gamma_2/6$.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \leq 3$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$. If $s_4 \geq 4$ we get
\( \epsilon_4 + 5\gamma_3 + \beta_2^{(4)} \leq h + \delta. \)

This contradicts \((45)\) for \(t = 4\) since \(\beta_2^{(4)} > \gamma_2/2.\)
Thus we are left with the possibilities \(r_2, r_3, r_5\) and \(s = 0, 2, 3, 5.\) Of course \(s = 1\) is impossible. Now if a case using a number of transfers with \(s = 2\) at the end of the list gives a coefficient below 2.008, then the corresponding case with the same transfers and having \(s = 3\) at the end of the list has an even lower coefficient. This is because in the corresponding average inequality we have to add \((\gamma_3 + \gamma_2 - \beta_2^{(4)})/l,\) where \(l\) is the length of the list. Therefore we do not list the results for \(s = 3\) when the corresponding average inequality for \(s = 2\) yields a coefficient below 2.008.

**Table 31 A.** \(K(1)\) for \(s = 0\) and \(s > 0\) in the main list in interval \(I_9,\)
Part I

<table>
<thead>
<tr>
<th>(r_5)</th>
<th>(r_3)</th>
<th>(r_2)</th>
<th>Average inequality</th>
<th>(s)</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0  0  0</td>
<td></td>
<td></td>
<td>(\epsilon_4 + \gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta)</td>
<td>s=0</td>
<td>1.5</td>
</tr>
<tr>
<td>0  0  1</td>
<td></td>
<td></td>
<td>(\epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta)</td>
<td>s=0</td>
<td>2.0</td>
</tr>
<tr>
<td>0  1  0</td>
<td></td>
<td></td>
<td>(\epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta)</td>
<td>s=0</td>
<td>1.28</td>
</tr>
<tr>
<td>0  1  1</td>
<td></td>
<td></td>
<td>(\epsilon_4 + 8\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta)</td>
<td>s=0</td>
<td><strong>2.082</strong></td>
</tr>
<tr>
<td>1  0  0</td>
<td></td>
<td></td>
<td>(\epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta)</td>
<td>s=0</td>
<td>1.46</td>
</tr>
<tr>
<td>1  0  1</td>
<td></td>
<td></td>
<td>(\epsilon_4 + 10\gamma_3/3 + (4\gamma_2 - 4\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta)</td>
<td>s=0</td>
<td><strong>2.13</strong></td>
</tr>
<tr>
<td>1  1  0</td>
<td></td>
<td></td>
<td>(\epsilon_4 + 11\gamma_3/3 + (5\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta)</td>
<td>s=0</td>
<td>1.71</td>
</tr>
<tr>
<td>1  1  1</td>
<td></td>
<td></td>
<td>(\epsilon_4 + 14\gamma_3/4 + (6\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta)</td>
<td>s=0</td>
<td><strong>2.17</strong></td>
</tr>
</tbody>
</table>

**Table 31 B.** \(K(1)\) for \(s = 2, 3, 5\) and \(s > 0\) in the main list in interval \(I_9,\)
Part II
There are only three cases left for $K(1)$: $r_2, r_3$ and $s = 0$, $r_2, r_5$ and $s = 0$ and $r_2, r_3, r_5$ and $s = 0$. All three cases have $s = 0$. We now have a closer look at the last case $r_2, r_3, r_5$ and $s = 0$ and go through the different orderings.

**A)** $K(1)$: $r_2, r_3, r_5$ and $s = 0$

- $\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta$
- $\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 3] \gamma_1 \leq h + \delta$
- $\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5] \gamma_1 \leq h + \delta$
- $\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta$

Using the weights 0, 1, 2 and 1 yields

$$\epsilon_4 + 17\gamma_3/4 + (8\gamma_2 - 9\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$$

giving a coefficient 1.94 < 2.008 according to (41) and (43).

**B)** $K(1)$: $r_2, r_5, r_3$ and $s = 0$

- $\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta$
- $\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + 1\beta_1^{(3)} - [1, 2] \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5] \gamma_1 \leq h + \delta$
- $\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] \gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 3] \gamma_1 \leq h + \delta$
Using the weights 1, 0, 1 and 0 yields

\[ \epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.46 < 2.008 according to (41) and (43).

C) \( K(1) \): \( r_3, r_2, r_5 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta \\
\end{align*}
\]

This case can not be excluded as easily as the other so we have to wait until the end of the section.

D) \( K(1) \): \( r_3, r_5, r_2 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 &+ 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 3]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 \leq h + \delta \\
\end{align*}
\]

Using the weights 0, 1, 0 and 1 yields

\[ \epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.46 < 2.008 according to (41) and (43).

E) \( K(1) \): \( r_5, r_2, r_3 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 &+ 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta \\
\end{align*}
\]

This case can not be excluded as easily as the other so we have to wait until the end of the section.

F) \( K(1) \): \( r_5, r_3, r_2 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 &+ 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta \\
\end{align*}
\]
Using the weights 1, 2, 1 and 0 yields

\[ \epsilon_4 + 17\gamma_3/4 + (8\gamma_2 - 9\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \]

giving a coefficient 1.94 < 2.008 according to (41) and (43). Thus there are only two inequality systems left in the third case.

Now we are going to study the main list \( K(0) \). We can conclude that only \( s = 1 \) and \( s = 4 \) are possible. If we had \( s = 2 \) we must have \( s = 2 \) in the list for \( K(1) \), too, but we do not. The same is true for \( s = 5 \). The transfer \( r_6 \) is excluded in the same way as for \( K(1) \). Thus we are left with \( r_1, r_2, r_3, r_4, r_5 \) with \( s = 1 \) or \( s = 4 \). We also see that \( r_3 \) and \( s = 1 \) cannot occur in the same main list since \( \gamma_2 - \beta_2^{(4)} \leq 0 \).

### Table 32 A. Mainlist for \( s = 1, 4 \) in interval \( J_9 \), Part I, \( r_5 = r_4 = 0 \)

<table>
<thead>
<tr>
<th>( r_3 )</th>
<th>( r_2 )</th>
<th>( r_1 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=1</td>
<td>1.89</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=4</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=1</td>
<td>1.89</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 7\gamma_3/2 + (4\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>2.004*</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 5\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=1</td>
<td>1.89*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1.62</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 5\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=1</td>
<td>1.89*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \epsilon_4 + 10\gamma_3/3 + (6\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>2.41</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 9\gamma_3/2 + (6\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=4</td>
<td>1.13</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.89</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 12\gamma_3/3 + (8\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=4</td>
<td>1.56</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 14\gamma_3/4 + (9\gamma_2 - 10\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=4</td>
<td>2.17</td>
<td></td>
</tr>
</tbody>
</table>

### Table 32 B. Mainlist for \( s = 1, 4 \) in interval \( J_9 \), Part II, \( r_4 = 1, r_5 = 0 \)
<table>
<thead>
<tr>
<th>$r_3$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeffi.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 7\gamma_3/2 + (4\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>2.004*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>s=4</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 7\gamma_3/2 + (4\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>2.004*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 7\gamma_3/2 + (4\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>2.004*</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 10\gamma_3/3 + (6\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=1</td>
<td>2.41</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.62</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 10\gamma_3/3 + (6\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=1</td>
<td>2.41</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 10\gamma_3/3 + (6\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>2.41</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/2 + (6\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.13</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.89</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 12\gamma_3/3 + (8\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.56</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 14\gamma_3/4 + (9\gamma_2 - 10\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>2.17</td>
<td>4</td>
</tr>
</tbody>
</table>

In the main list we cannot have both $r_5 = 1$ and $r_3 = 1$, since they would imply

$$\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} \leq h + \delta$$

and

$$\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} \leq h + \delta.$$

Together this gives

$$\epsilon_4 + 5\gamma_3 + \gamma_2/2 \leq h + \delta$$

contradicting (45) for $t = 4$. So we may exclude $r_3$ from the list below.
Table 32 C. Mainlist for $s = 1, 4$ in interval $I_9$, Part III, $r_3 = 0, r_5 = 1$

<table>
<thead>
<tr>
<th>$r_4$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>s</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.62</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/2 + (7\gamma_2 - 9\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.47</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>1.62</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 13\gamma_3/3 + (8\gamma_2 - 10\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.95*</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.93</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/4 + (10\gamma_2 - 12\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>1.72</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.93</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/4 + (10\gamma_2 - 12\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>2.25</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 13\gamma_3/3 + (8\gamma_2 - 10\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=1</td>
<td>1.95*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 11\gamma_3/2 + (7\gamma_2 - 9\beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=4</td>
<td>1.47</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 13\gamma_3/3 + (8\gamma_2 - 10\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=1</td>
<td>1.95*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 13\gamma_3/3 + (8\gamma_2 - 10\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.95*</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 16\gamma_3/4 + (10\gamma_2 - 12\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>2.25</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 14\gamma_3/3 + (9\gamma_2 - 11\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=4</td>
<td>1.72</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 16\gamma_3/4 + (10\gamma_2 - 12\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>2.25</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\epsilon_4 + 16\gamma_3/4 + (10\gamma_2 - 12\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=4</td>
<td>2.25</td>
<td>8</td>
</tr>
</tbody>
</table>

Thus we are left with ten cases. We now go through all of them.

1) $K(0)$: $r_1, r_2$ and $s = 4$. Here we get two orderings

A) $K(0)$: $r_1, r_2$ and $s = 4$

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta
$$
$$
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + 1\beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta.
$$

Here we have $\kappa_l = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 = 0$. If $[2, 4]$ is even then $2\beta_1^{(3)} + \beta_1^{(4)} + [1, 2]\gamma_1 = 0$, too and adding the two first lines yields:

$$
\epsilon_4 + 5\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
$$
giving a coefficient $1.89 < 2.008$ according to (71).

If $[2, 4]$ is odd then the last line reads

$$
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1/2 \leq h + \delta
$$
giving a coefficient $1.98 < 2.008$ according to (41) and (43).

B) $K(0)$: $r_2, r_1$ and $s = 4$

$$
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
$$
\[ 
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + -2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta. 
\]

Here we have \( \kappa_l = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 = 0 \). If \([2, 4]\) is even then
\[ 2\beta_1^{(3)} + \beta_1^{(4)} + [1, 2]\gamma_1 = 0, \text{ too and the first line reads} \]
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta \]
giving a coefficient 0.67 < 2.008 according to (41) and (43).
If \([2, 4]\) is odd then the first line reads
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1/2 \leq h + \delta \]
still giving a coefficient 1.33 < 2.008 according to (41) and (43).

2) \( K(0) \): \( r_4, r_1, r_2 \) and \( s = 4 \). Here we get two orderings
A) \( K(0) : r_4, r_1, r_2 \) and \( s = 4 \)
\[ 
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta. 
\]

Here we have \( \kappa_l = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 < 0 \). The last line then reads in two different ways:
\[ 
\epsilon_4 + 5\gamma_3 + 3\gamma_2 + 2\beta_2^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta. 
\]

Averging gives
\[ \epsilon_4 + 5\gamma_3 + (3\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
contradicting (45) for \( t = 4 \).

B) \( K(0) : r_4, r_2, r_1 \) and \( s = 4 \)
\[ 
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(3)} - 2\beta_1^{(4)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + -2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta. 
\]

Using the weights 1, 2, 0 and 0 yields
\[ \epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient $1.93 < 2.008$ according to (41) and (43).

3) $K(0)$: $r_1, r_2, r_3$ and $s = 4$. Here we get six orderings

A) $K(0)$: $r_1, r_2, r_3$ and $s = 4$

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(3)} - [1,2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2]\gamma_1 - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2,3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2,3]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2,4]\gamma_1 \leq h + \delta.
$$

Using the weights 1, 0, 2 and 1 yields

$$
\epsilon_4 + 15\gamma_3/4 + (10\gamma_2 - 11\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
$$

giving a coefficient $1.87 < 2.008$ according to (41) and (43).

B) $K(0)$: $r_1, r_3, r_2$ and $s = 4$

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(3)} - 3\beta_1^{(3)} - 2\beta_1^{(4)} + [2,3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2,3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} + \beta_1^{(3)} - [1,2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2,4]\gamma_1 \leq h + \delta.
$$

Using the weights 0, 1, 2 and 0 yields

$$
\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$

giving a coefficient $1.47 < 2.008$ according to (41) and (43).

C) $K(0)$: $r_2, r_1, r_3$ and $s = 4$

$$
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2]\gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2,3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2,3]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2,4]\gamma_1 \leq h + \delta.
$$

Here we have $\kappa_1 = 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2,4]\gamma_1 = 0$. Therefore $[2,4] = [2,4]$. If $[2,4]$ is even then $2\beta_1^{(4)} + \beta_1^{(3)} - [1,2]\gamma_1 = 0$, too and the first line reads

$$
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta
$$

giving a coefficient $0.67 < 2.008$ according to (41) and (43).

If $[2,4]$ is odd the first line reads

$$
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1/2 \leq h + \delta
$$
still giving a coefficient 1.33 < 2.008 according to (41) and (43).

D) $K(0)$: $r_2, r_3, r_1$ and $s = 4$

\begin{align*}
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}

Using the weights 0, 2, 1 and 0 yields

$$\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$$

giving a coefficient 1.47 < 2.008 according to (41) and (43).

E) $K(0)$: $r_3, r_1, r_2$ and $s = 4$

\begin{align*}
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}

Here we have $\kappa_l = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 = 0$. Therefore $[2, 4] = [2, 4]$. If $[2, 4]$ is even then $2\beta_1^{(3)} + \beta_1^{(4)} - [1, 2]\gamma_1 = 0$, too and the reduction of the constant term in line 3 is $\leq 0$ and we should have had a shorter list $l = 3$, a contradiction.

If $[2, 4]$ is odd then the last line reads

$$\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1/2 \leq h + \delta$$

giving a coefficient 1.35 < 2.008 according to (41) and (43).

F) $K(0)$: $r_3, r_2, r_1$ and $s = 4$

\begin{align*}
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}

Using the weights 1, 2, 0 and 1 yields

$$\epsilon_4 + 15\gamma_3/4 + (10\gamma_2 - 11\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$$

giving a coefficient 1.87 < 2.008 according to (41) and (43).

4) $K(0)$: $r_4, r_1, r_2, r_3$ and $s = 4$. Here we get six orderings
A) $K(0)$: $r_4, r_1, r_2, r_3$ and $s = 4$

$$
\begin{align*}
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(3)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}
$$

Using the weights $0, 1, 0, 2$ and $1$ yields

$$
\epsilon_4 + 15\gamma_3/4 + (10\gamma_2 - 11\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
$$

giving a coefficient $1.87 < 2.008$ according to (41) and (43).

B) $K(0)$: $r_4, r_1, r_2, r_3$ and $s = 4$

$$
\begin{align*}
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}
$$

Using the weights $0, 0, 1, 2$ and $0$ yields

$$
\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$

giving a coefficient $1.47 < 2.008$ according to (41) and (43).

C) $K(0)$: $r_4, r_2, r_1, r_3$ and $s = 4$

$$
\begin{align*}
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [1, 2]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}
$$

Using the weights $1, 2, 0, 0$ and $0$ yields

$$
\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$

giving a coefficient $1.93 < 2.008$ according to (41) and (43).
D) $K(0)$: $r_4, r_2, r_3, r_1$ and $s = 4$

\[
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\beta_3 + 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta.
\]

Using the weights 1, 2, 0, 0 and 0 yields

\[
\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]
giving a coefficient 1.93 < 2.008 according to (41) and (43).

E) $K(0)$: $r_4, r_3, r_1, r_2$ and $s = 4$

\[
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta.
\]

Using the weights 1, 2, 1, 0 and 0 yields

\[
\epsilon_4 + 15\gamma_3/3 + (10\gamma_2 - 11\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
\]
giving a coefficient 1.87 < 2.008 according to (41) and (43).

F) $K(0)$: $r_4, r_3, r_2, r_1$ and $s = 4$

\[
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta.
\]

Using the weights 0, 1, 2, 0 and 1 yields

\[
\epsilon_4 + 15\gamma_3/4 + (10\gamma_2 - 11\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
\]
giving a coefficient 1.87 < 2.008 according to (41) and (43).

5) $K(0)$: $r_2, r_4$ and $s = 1$. Here we get two orderings
Using the weights 1, 2, and 0 yields
\[
\epsilon_4 = \gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4] \gamma_1 \leq h + \delta 
\]
\[
\epsilon_4 = 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} + [2, 4] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta 
\]
\[
\epsilon_4 = 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - \beta_1^{(4)} \leq h + \delta. 
\]
Using the weights 1, 2, and 0 yields
\[
\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta 
\]
giving a coefficient 1.93 < 2.008 according to (41) and (43).

B) \(K(0): r_2, r_4\) and \(s = 1\)
\[
\epsilon_4 = 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta 
\]
\[
\epsilon_4 = 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4] \gamma_1 \leq h + \delta 
\]
\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4] \gamma_1 - \beta_1^{(4)} \leq h + \delta. 
\]
Here we have to include what we know about \(K(1)\), (three cases). If \(K(1)\) is represented by \(r_2, r_3\) and \(s = 0\) we know that \(r_2\) comes first and we get an additional contribution. The amount is \(\gamma_2 - \beta_2^{(4)}\). Thus the original average inequality for \(K(1)\)
\[
\epsilon_4 + 8\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta 
\]
turns into
\[
\epsilon_4 + 8\gamma_3/3 + (4\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta 
\]
giving a coefficient 1.84 < 2.008 according to (41) and (43). In the second case for the \(K(1)\)-list, \(r_2, r_5\) and \(s = 0\), the average inequality
\[
\epsilon_4 + 10\gamma_3/3 + (4\gamma_2 - 4\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta 
\]
turns into
\[
\epsilon_4 + 10\gamma_3/3 + (5\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta 
\]
giving a coefficient 1.88 < 2.008 according to (41) and (43). When \(K(1)\) is represented by \(r_2, r_3, r_5\) and \(s = 0\) we have already shown that \(r_2\) cannot be first. Thus this case is impossible.

6) \(K(0): r_1, r_2, r_4\) and \(s = 1\). Here we get two orderings. We have to combine the main list with the \(K(1)\)-list.

A) \(K(0): r_1, r_4, r_2\) and \(s = 1\)
\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta 
\]
\[
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4] \gamma_1 \leq h + \delta 
\]
\[
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} + [2, 4] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta 
\]
\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - \beta_1^{(4)} \leq h + \delta. 
\]
In the first case for $K(1)$ where $K(1)$ is represented by $r_3, r_2$ and $s = 0$ the list reads

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(4)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 0, 0, 1, 0, and 1, 1, 0 yields

\[
\epsilon_4 + 10\gamma_3/3 + (5\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]
giving a coefficient $1.88 < 2.008$ according to (41) and (43). The other ordering for the $K(1)$ list gives

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(4)} - [2, 3]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 0, 0, 1, 0, and 1, 0, 0 yields

\[
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient $1.70 < 2.008$ according to (41) and (43).

The other ordering for the main list gives the following.

B) $K(0)$: $r_1, r_2, r_4$ and $s = 1$

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} + [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(4)} + [2, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(4)} + \gamma_1 & \leq h + \delta.
\end{align*}
\]

In the first case for $K(1)$ where $K(1)$ is represented by $r_2, r_3$ and $s = 0$ the list reads

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(4)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(4)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 0, 0, 0, 1 and 1, 0, 0 yields

\[
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient $1.70 < 2.008$ according to (41) and (43).
The other ordering for the $K(1)$-list gives

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [2, 1] \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(4)} + [2, 3] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 \leq h + \delta.
$$

Using the weights 0, 0, 1, 0 and 0, 1, 1 yields

$$
\epsilon_4 + 10\gamma_3/3 + (5\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$

giving a coefficient 1.88 < 2.008 according to (41) and (43).

Now we choose the second alternative for the $K(1)$-list $r_2, r_5$ and $s = 0$.

A) $K(0)$: $r_1, r_4, r_2$ and $s = 1$

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta
$$

$$
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} + [2, 4] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.
$$

The first ordering for the $K(1)$-list gives

$$
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(4)} + [3, 5] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(4)} - [3, 5] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 \leq h + \delta.
$$

Using the weights 0, 1, 0, 0 and 0, 1, 0 yields

$$
\epsilon_4 + 8\gamma_3/2 + (4\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
$$

giving a coefficient 1.78 < 2.008 according to (71). The other ordering for the $K(1)$ list gives

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [1, 2] \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(4)} + [3, 5] \gamma_1 \leq h + \delta
$$

$$
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] \gamma_1 \leq h + \delta.
$$

Using the weights 0, 0, 1, 0 and 1, 0, 0 yields

$$
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
$$

giving a coefficient 1.70 < 2.008 according to (41) and (43).

The other ordering for the main list gives the following.
B) $K(0)$: $r_1, r_2, r_4$ and $s = 1$

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(3)} - 2\gamma_1 - 4\beta_2^{(4)} - 2\gamma_1 - 2\beta_1^{(3)} + [2, 4] \gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4] \gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.
$$

The first ordering for the $K(1)$-list gives

$$
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5] \gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 \leq h + \delta.
$$

Using the weights 0, 0, 1, 0 and 0, 0, 1 yields

$$
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
$$
giving a coefficient $1.70 < 2.008$ according to (41) and (43).

The other ordering for the $K(1)$-list gives

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - 5\beta_2^{(4)} + \gamma_1 + 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5] \gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5] \gamma_1 \leq h + \delta.
$$

Using the weights 0, 0, 1, 0 and 0, 1, 0 yields

$$
\epsilon_4 + 8\gamma_3/2 + (4\gamma_2 - 5\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
$$
giving a coefficient $1.78 < 2.008$ according to (41) and (43).

Now we choose the third and last alternative for the $K(1)$ list $r_3, r_2, r_5$ and $s = 0$. Only two orderings for this list are possible.

A) $K(0)$: $r_1, r_4, r_2$ and $s = 1$

$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(3)} - 2\gamma_1 - 2\beta_1^{(3)} + [2, 4] \gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.
$$

In the first case for $K(1)$ where $K(1)$ is represented by $r_3, r_2, r_5$ and $s = 0$ the list reads

$$
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3] \gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5] \gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] \gamma_1 \leq h + \delta.
$$
Using the weights 0, 0, 1, 0 and 0, 1, 1, 0 yields
\[ \epsilon_4 + 12\gamma_3/3 + (6\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.92 < 2.008 according to (71).
The other ordering for the main list gives
\[ \epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(4)} + [3, 5]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(4)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta. \]
Using the weights 0, 1, 0, 0 and 0, 0, 0, 1 yields
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.70 < 2.008 according to (41) and (43).
The other ordering for the first case for \(K(1)\) where \(K(1)\) is represented by \(r_3, r_2, r_5\) and \(s = 0\) the list reads
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [1, 2]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(4)} + [3, 5]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 \leq h + \delta. \]
Using the weights 0, 0, 0, 1 and 1, 0, 0 yields
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.70 < 2.008 according to (41) and (43).
The other ordering for the \(K(1)\)-list gives
\[ \epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(4)} + [3, 5]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(4)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta. \]
Averaging using the weights 0, 0, 1, 0 and 0, 0, 1, 1 we get
\[ \epsilon_4 + 10\gamma_3/3 + (5\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.88 < 2.008 according to (41) and (43).

7) \(K(0): r_1, r_2, r_5\) and \(s = 4\). Here we get six orderings
A) \(K(0): r_1, r_2, r_5\) and \(s = 4\)

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(3)} + \beta_1^{(4)} - [1, 2]\gamma_1 - 5\beta_1^{(3)} - 3\beta_1^{(4)} + [3, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 0, 1, 0, and 1 yields
\[ \epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.62 < 2.008 according to (41) and (43).

B) \(K(0): r_1, r_2, r_2\) and \(s = 4\)

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Here we have \(\kappa_l = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 = 0\). Therefore \([2, 4] = [2, 4]\). If \([2, 4]\) is even then \(2\beta_1^{(3)} + \beta_1^{(4)} - [1, 2]\gamma_1 = 0\), too and the reduction of the constant term in line 3 is \(\leq 0\) and we should have had a shorter list \(l = 3\), a contradiction. If \([2, 4]\) is odd then the last line reads
\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.98 < 2.008 according to (41) and (43).

C) \(K(0): r_2, r_1, r_5\) and \(s = 4\)

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(3)} + \beta_1^{(4)} - [1, 2]\gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(3)} - 5\beta_1^{(3)} - 3\beta_1^{(4)} + [3, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(2)} + [2, 4]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Here we have \(\kappa_l = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 = 0\). Therefore \([2, 4] = [2, 4]\). If \([2, 4]\) is even then \(2\beta_1^{(3)} + \beta_1^{(4)} - [1, 2]\gamma_1 = 0\), too and the first line reads
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta \]
giving a coefficient 0.67 < 2.008 according to (41) and (43).
If \([2, 4]\) is odd then the first line reads
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1/2 \leq h + \delta \]
still giving a coefficient 0.67 < 2.008 according to (41) and (43).

D) \(K(0): r_2, r_5, r_1\) and \(s = 4\)

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 &- 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 &\leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 &\leq h + \delta.
\end{align*}
\]

Here we have \(\kappa_4 = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 = 0\). Therefore \([2, 4] = [2, 4]\). If \([2, 4]\) is even then \(2\beta_1^{(3)} + \beta_1^{(4)} - [1, 2]\gamma_1 = 0\), too and the first line reads
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 \leq h + \delta \]
giving a coefficient 0.67 < 2.008 according to (41) and (43).
If \([2, 4]\) is odd then the first line reads
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1/2 \leq h + \delta \]
still giving a coefficient 1.33 < 2.008 according to (41) and (43).

E) \(K(0): r_5, r_1, r_2\) and \(s = 4\)

\[
\begin{align*}
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \gamma_1 &- 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - \beta_1^{(4)} &\leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} - [1, 2]\gamma_1 &\leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 &\leq h + \delta.
\end{align*}
\]

Here we have \(\kappa_4 = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 = 0\). Therefore \([2, 4] = [2, 4]\). If \([2, 4]\) is even then \(2\beta_1^{(3)} + \beta_1^{(4)} - [1, 2]\gamma_1 = 0\), too and the reduction of the constant term in line 3 is \(\leq 0\) and we should have had a shorter list \(l = 3\), a contradiction.
If \([2, 4]\) is odd then the last line reads
\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.98 < 2.008 according to (41) and (43).

\[ K(0): \ r_5, r_2, r_1 \text{ and } s = 4 \]

\[ \epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 - \beta_1^{(4)} \leq h + \delta \]

\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]

Using the weights 0, 1, 0 and 1 yields

\[ \epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]

8) \( K(0): \ r_4, r_1, r_2, r_5 \text{ and } s = 4 \). Here again we get six orderings

A) \( K(0): \ r_4, r_1, r_2, r_5 \text{ and } s = 4 \)

\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(4)} \leq h + \delta \]

\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]

Using the weights 0, 0, 1, 0 and 1 yields

\[ \epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/4 + \gamma_1/2 \leq h + \delta \]

B) \( K(0): \ r_4, r_1, r_5, r_2 \text{ and } s = 4 \)

\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(4)} \leq h + \delta \]

\[ \epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]

\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \]

Using the weights 0, 0, 1, 2 and 1 yields

\[ \epsilon_4 + 17\gamma_3/4 + (11\gamma_2 - 13\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \]

giving a coefficient 1.94 < 2.008 according to (41) and (43).
C) $K(0)$: $r_4, r_2, r_1, r_5$ and $s = 4$

\[\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + 3\beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} \leq h + \delta\]
\[\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta.\]

Using the weights 1, 2, 0, 0 and 0 yields

\[\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\]

giving a coefficient 1.93 < 2.008 according to (41) and (43).

D) $K(0)$: $r_4, r_2, r_5, r_1$ and $s = 4$

\[\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] - \beta_1^{(4)} \leq h + \delta\]
\[\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta.\]

Using the weights 1, 2, 0, 0 and 0 yields

\[\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\]

giving a coefficient 1.93 < 2.008 according to (41) and (43).

E) $K(0)$: $r_4, r_5, r_1, r_2$ and $s = 4$

\[\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] - \beta_1^{(4)} \leq h + \delta\]
\[\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta\]
\[\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta.\]

Since $\kappa_t = 4\beta_1^{(3)} + 2\beta_1^{(4)} - [2, 4]\gamma_1 < 0$ in the last line this line reads

\[\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [2, 1]\gamma_1 \leq h + \delta.\]

Combining this line with the last one from the list we get

\[\epsilon_4 + 5\gamma_3 + \gamma_1/2 \leq h + \delta\]
giving a coefficient $1.98 < 2.008$ according to (41) and (43).

9) $K(0)$: $r_2, r_4, r_5$ and $s = 1$. Here we get six orderings. We know that $\beta_1^{(4)} = 0$

A) $K(0)$: $r_2, r_4, r_5$ and $s = 1$

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} + [1, 2] &\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 + 3\beta_1^{(3)} + [3, 5] &\gamma_1 \leq h + \delta  \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(3)} - [2, 4] &\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} - [1, 2] &\gamma_1 \leq h + \delta  \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_1^{(4)} + \gamma_1 - 2\beta_1^{(4)} - 3\beta_1^{(3)} + [2, 4] &\gamma_1 \leq h + \delta.
\end{align*}
\]

Using the weights 0, 0, 3 and 1 yields

\[
\begin{align*}
\epsilon_4 + 20\gamma_3/4 + (13\gamma_2 - 16\beta_2^{(4)})/4 + \gamma_1/3 &\geq h + \delta
\end{align*}
\]

giving a coefficient $1.80 < 2.008$ according to (41) and (43).

B) $K(0)$: $r_2, r_5, r_4$ and $s = 1$

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} + [1, 2] &\gamma_1 \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(3)} - [1, 2] &\gamma_1 - 3\beta_1^{(3)} + [3, 5] &\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5] &\gamma_1 - 2\beta_1^{(3)} + [2, 4] &\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(3)} - [2, 4] &\gamma_1 \leq h + \delta.
\end{align*}
\]

Using the weights 1, 0, 1 and 0 yields

\[
\begin{align*}
\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 &\geq h + \delta
\end{align*}
\]

giving a coefficient $1.62 < 2.008$ according to (41) and (43).
C) \( K(0): r_4, r_2, r_5 \) and \( s = 1 \)

\[
\begin{align*}
\epsilon_4 + 5\gamma_3 &+ 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 &+ 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 6\gamma_3 &+ 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 &+ \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 \leq h + \delta.
\end{align*}
\]

Using the weights 0, 2, 1 and 0 yields

\[
\epsilon_4 + 12\gamma_3/3 + (8\gamma_2 - 9\beta_2^{(4)})/4 + \gamma_1/3 \leq h + \delta
\]

giving a coefficient \( 1.56 < 2.008 \) according to (41) and (43).

D) \( K(0): r_4, r_5, r_2 \) and \( s = 1 \)

\[
\begin{align*}
\epsilon_4 + 5\gamma_3 &+ 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 6\gamma_3 &+ 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 &+ 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 &+ \gamma_2 - \beta_2^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta.
\end{align*}
\]

Using the weights 0, 1, 0 and 1 yields

\[
\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]

giving a coefficient \( 1.62 < 2.008 \) according to (41) and (43).

E) \( K(0): r_5, r_2, r_4 \) and \( s = 1 \)

\[
\begin{align*}
\epsilon_4 + 6\gamma_3 &+ 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 &+ 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 &+ 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 &+ \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 \leq h + \delta.
\end{align*}
\]

Using the weights 0, 1, 2 and 0 yields

\[
\epsilon_4 + 13\gamma_3/3 + (8\gamma_2 - 10\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]

giving a coefficient \( 1.95 < 2.008 \) according to (71).

F) \( K(0): r_5, r_4, r_2 \) and \( s = 1 \)

\[
\begin{align*}
\epsilon_4 + 6\gamma_3 &+ 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 + 5\gamma_3 &+ 3\gamma_2 - 4\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 &+ 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 + 2\gamma_3 &+ \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta.
\end{align*}
\]
Using the weights 1, 0, 3 and 0 yields
\[ \epsilon_4 + 15 \gamma_3/4 + (10 \gamma_2 - 11 \beta_2^{(4)})/4 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.87 < 2.008 according to (41) and (43).

10) \( K(0) \): \( r_1, r_2, r_4, r_5 \) and \( s = 1 \). Here we get six orderings.

A) \( K(0) \): \( r_1, r_2, r_4, r_5 \) and \( s = 1 \)

\[ \begin{align*}
\epsilon_4 + 2 \gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 3 \gamma_3 + 2 \gamma_2 - 2 \beta_2^{(4)} + \beta_1^{(4)} - 2 \beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 5 \gamma_3 + 3 \gamma_2 - 4 \beta_2^{(4)} + 2 \beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 4 \beta_1^{(4)} - 2 \beta_1^{(3)} + [2, 4] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 6 \gamma_3 + 4 \gamma_2 - 5 \beta_2^{(4)} + 4 \beta_1^{(4)} + 2 \beta_1^{(3)} - [2, 4] \gamma_1 - 5 \beta_1^{(4)} - 3 \beta_1^{(3)} + [3, 5] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 2 \gamma_3 + \gamma_2 - \beta_2^{(4)} + 5 \beta_1^{(4)} + 3 \beta_1^{(3)} - [3, 5] \gamma_1 - \beta_1^{(4)} + \gamma_1 & \leq h + \delta.
\end{align*} \]

Since the main list contains \( r_5 \) the list for \( K(1) \) also has to contain \( r_5 \) and we have two choices. We start with the case where \( K(1) \) is represented by \( r_2, r_5 \) and \( s = 0 \). Since \( r_2 \) comes before \( r_5 \) in the main list we have the same order in the list for \( K(1) \). This list now reads

\[ \begin{align*}
\epsilon_4 + 3 \gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2 \beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 6 \gamma_3 + 3 \gamma_2 - 4 \beta_2^{(4)} + 2 \beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 5 \beta_1^{(4)} - 3 \beta_1^{(3)} + [3, 5] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 5 \beta_1^{(4)} + 3 \beta_1^{(3)} - [3, 5] \gamma_1 & \leq h + \delta.
\end{align*} \]

Using the weights 0, 0, 1, 1 and 0, 1, 0 yields
\[ \epsilon_4 + 14 \gamma_3/3 + (8 \gamma_2 - 10 \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.95 < 2.008 according to (41) and (43).

Now we consider the second choice for \( K(1) \), namely \( r_3, r_2, r_5 \) and \( s = 0 \). We have got two different orderings for the \( K(1) \)-list. Only one of them fits here:

\[ \begin{align*}
\epsilon_4 + 4 \gamma_3 + 2 \gamma_2 - 2 \beta_2^{(4)} + \gamma_1 - 3 \beta_1^{(4)} - 2 \beta_1^{(3)} + [2, 3] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 3 \gamma_3 + \gamma_2 - \beta_2^{(4)} + 3 \beta_1^{(4)} + 2 \beta_1^{(3)} - [2, 3] \gamma_1 - 2 \beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 6 \gamma_3 + 3 \gamma_2 - 4 \beta_2^{(4)} + 2 \beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 5 \beta_1^{(4)} - 3 \beta_1^{(3)} + [3, 5] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 5 \beta_1^{(4)} + 3 \beta_1^{(3)} - [3, 5] \gamma_1 & \leq h + \delta.
\end{align*} \]

Using the weights 0, 0, 1, 0 and 0, 1, 0, 0 yields
\[ \epsilon_4 + 9 \gamma_3/2 + (5 \gamma_2 - 6 \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.44 < 2.008 according to (41) and (43).
B) $K(0)$: $r_1, r_2, r_5, r_4$ and $s = 1$

$$
e_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta$$
$$
e_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta$$
$$
e_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta$$
$$
e_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta$$
$$
e_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.
$$

Using the weights 0, 1, 0, 1 and 0 yields

$$
e_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$$

giving a coefficient $1.62 < 2.008$ according to (41) and (43).

C) $K(0)$: $r_1, r_4, r_2, r_5$ and $s = 1$

$$
e_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta$$
$$
e_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta$$
$$
e_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta$$
$$
e_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta$$
$$
e_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.$$

Using the weights 0, 0, 1, 2 and 1 yields

$$
e_4 + 17\gamma_3/4 + (11\gamma_2 - 13\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$$

giving a coefficient $1.94 < 2.008$ according to (41) and (43).

D) $K(0)$: $r_1, r_4, r_5, r_2$ and $s = 1$

$$
e_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta$$
$$
e_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \beta_1^{(4)} - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 \leq h + \delta$$
$$
e_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta$$
$$
e_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta$$
$$
e_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.$$

Using the weights 0, 1, 0, 1 and 0 yields

$$
e_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$$

giving a coefficient $1.62 < 2.008$ according to (41) and (43).
E) $K(0)$: $r_1, r_5, r_2, r_4$ and $s = 1$

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 4]\gamma_1 - \beta_1^{(4)} + \gamma_1 & \leq h + \delta .
\end{align*}
\]

Using the weights $0, 1, 2, 1$ and $0$ yields

\[
\epsilon_4 + 17\gamma_3/4 + (11\gamma_2 - 13\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
\]
giving a coefficient $1.94 < 2.008$ according to (41) and (43).

F) $K(0)$: $r_1, r_5, r_4, r_2$ and $s = 1$

\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 4\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 4]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 4\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 1]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} + \gamma_1 & \leq h + \delta .
\end{align*}
\]

Since the main list contains $r_5$ the list for $K(1)$ also has to contain $r_5$ and we have two choices. We start with the case where $K(1)$ is represented by $r_2, r_5$ and $s = 0$. Since $r_5$ comes before $r_2$ in the main list we have the same order in the list for $K(1)$. This list now reads

\[
\begin{align*}
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [5, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [5, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [2, 1]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [2, 1]\gamma_1 & \leq h + \delta .
\end{align*}
\]

Using the weights $0, 0, 0, 1, 0$ and $1, 1, 0$ yields

\[
\epsilon_4 + 12\gamma_3/3 + (6\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]
giving a coefficient $1.92 < 2.008$ according to (71).

Now we consider the second choice for $K(1)$, namely $r_5, r_2, r_3$ and $s = 0$. We have got two different orderings for the $K(1)$-list. Only one of them fits here:

\[
\begin{align*}
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 4\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 & \leq h + \delta .
\end{align*}
\]
Using the weights 0, 0, 1, 0, 0 and 0, 0, 1, 0 yields
\[ \epsilon_4 + 9\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.44 < 2.008 according to (41) and (43). Thus we finished interval \( I_9 \).

### 12.11 Interval \( I_{10} \)

Again we start considering the key number
\[ M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1 \]
and try to find a representation for \( M(1) \) and all the numbers in the inverall \( K(1) = [M(1) - \gamma_1 + 1, M(1)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(1) \). Also here it turns out that only very few transfers are possible. No transfers from \( D^* \) and \( E \) can be used since \( s > 0 \).

In interval \( I_{10} \) we have \( \frac{3}{4} \gamma_2 < \beta_2^{(4)} < \frac{4}{5} \gamma_2 \). Therefore we have
\[ \gamma_2 > 5\beta_2^{(4)} - 3\gamma_2 > \beta_2^{(4)} > 6\beta_2^{(4)} - 4\gamma_2 > 2\beta_2^{(4)} - \gamma_2 > 3\beta_2^{(4)} - 2\gamma_2 > 4\beta_2^{(4)} - 3\gamma_2 > 0 \]
and only \( r_2, r_3, r_4 \) and \( r_6 \) from \( A \) may be used. If \( r_4 \) is used we have
\[ \epsilon_4 + 5\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} \leq h + \delta. \]

This contradicts (45) for \( t = 4 \) since \( 3\gamma_2 - 3\beta_2^{(4)} > \gamma_2/2 \).

Now we assume that \( (s_2, s_3, s_4) \in B^* \) is used. If \( s_4 \leq 3 \) the second term will be \( \geq \gamma_2 \) which is impossible since \( s > 0 \). If \( s_4 \geq 4 \) we get
\[ \epsilon_4 + 5\gamma_3 + \beta_2^{(4)} \leq h + \delta. \]

This contradicts (45) for \( t = 4 \) since \( \beta_2^{(4)} > \gamma_2/2 \).

Thus we are left with the possibilities \( r_2, r_3, r_6 \) and \( s = 0, 2, 3, 6 \). Of course \( s = 1 \) and \( s = 5 \) are impossible. Now \( s = 3 \) is impossible in \( K(1) \). The corresponding transfer could be used twice at the end of the list since
\[ \beta_2^{(4)} - (3\beta_2^{(4)} - 2\gamma_2) > 3\beta_2^{(4)} - 2\gamma_2. \]
We now show how to exclude the second case. We go through all the six

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\(r_6\) & \(r_3\) & \(r_2\) & Average inequality & \(s\) & Coeff. \\
\hline
0 & 0 & 0 & \(\epsilon_4 + \gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta\) & s=0 & 1.33 \\
0 & 0 & 1 & \(\epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta\) & s=0 & 2.0 \\
0 & 1 & 0 & \(\epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=0 & 1.33 \\
0 & 1 & 1 & \(\epsilon_4 + 8\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\) & s=0 & 2.18 \\
1 & 0 & 0 & \(\epsilon_4 + 8\gamma_3/2 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=0 & 1.21 \\
1 & 0 & 1 & \(\epsilon_4 + 11\gamma_3/3 + (5\gamma_2 - 5\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\) & s=0 & 1.93 \\
1 & 1 & 0 & \(\epsilon_4 + 12\gamma_3/3 + (6\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\) & s=0 & 1.60 \\
1 & 1 & 1 & \(\epsilon_4 + 15\gamma_3/4 + (7\gamma_2 - 7\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta\) & s=0 & 2.10 \\
\hline
\end{tabular}
\caption{K(1) for \(s = 0\) and \(s > 0\) in the main list in interval \(I_{10}\), Part I}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\(r_6\) & \(r_3\) & \(r_2\) & Average inequality & \(s\) & Coeff. \\
\hline
0 & 0 & 0 & \(\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta\) & s=2 & 1.44 \\
0 & 0 & 1 & \(\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta\) & s=2 & 1.44 \\
0 & 1 & 0 & \(\epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=2 & 1.69 \\
0 & 1 & 1 & \(\epsilon_4 + 7\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=2 & 1.69 \\
1 & 0 & 0 & \(\epsilon_4 + 10\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=2 & 1.68 \\
1 & 0 & 1 & \(\epsilon_4 + 10\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=2 & 1.68 \\
1 & 1 & 0 & \(\epsilon_4 + 14\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\) & s=2 & 1.85 \\
1 & 1 & 1 & \(\epsilon_4 + 14\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\) & s=2 & 1.85 \\
\hline
\end{tabular}
\caption{K(1) for \(s = 2\) and \(s > 0\) in the main list in interval \(I_{10}\), Part II}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\(r_6\) & \(r_3\) & \(r_2\) & Average inequality & \(s\) & Coeff. \\
\hline
0 & 0 & 0 & \(\epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 \leq h + \delta\) & s=6 & 0.96 \\
0 & 0 & 1 & \(\epsilon_4 + 10\gamma_3/2 + (3\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=6 & 1.68 \\
0 & 1 & 0 & \(\epsilon_4 + 11\gamma_3/2 + (6\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=6 & 1.29 \\
0 & 1 & 1 & \(\epsilon_4 + 14\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\) & s=6 & 1.85 \\
1 & 0 & 0 & \(\epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 \leq h + \delta\) & s=6 & 0.96 \\
1 & 0 & 1 & \(\epsilon_4 + 10\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=6 & 1.68 \\
1 & 1 & 0 & \(\epsilon_4 + 11\gamma_3/2 + (6\gamma_2 - 7\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\) & s=6 & 1.29 \\
1 & 1 & 1 & \(\epsilon_4 + 14\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta\) & s=6 & 1.85 \\
\hline
\end{tabular}
\caption{K(1) for \(s = 6\) and \(s > 0\) in the main list in interval \(I_{10}\), Part III}
\end{table}

Thus we only get two choices for \(K(1)\), \(r_2, r_3\) and \(s = 0\) and \(r_2, r_3, r_6\) and \(s = 0\). We now show how to exclude the second case. We go through all the six orderings.
A) \( K(1) : r_2, r_3, r_6 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 - 6\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 6] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 6] \gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 0, 0, 2 and 1 yields

\[
\epsilon_4 + 15\gamma_3/3 + (8\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]
giving a coefficient 1.57 < 2.008 according to (41) and (43).

B) \( K(1) : r_2, r_6, r_3 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 6\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 6] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 1, 1, 2 and 0 yields

\[
\epsilon_4 + 18\gamma_3/4 + (9\gamma_2 - 10\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta
\]
giving a coefficient 2.0 < 2.008 according to (41) and (43).

C) \( K(1) : r_3, r_2, r_6 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [1, 2] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 6\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 6] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 6\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 6] \gamma_1 & \leq h + \delta.
\end{align*}
\]

Using the weights 2, 0, 0 and 1 yields

\[
\epsilon_4 + 9\gamma_3/3 + (4\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]
giving a coefficient 1.76 < 2.008 according to (41) and (43).

D) \( K(1) : r_3, r_6, r_2 \) and \( s = 0 \)

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 - 6\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 6] \gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 6\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 6] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 & \leq h + \delta.
\end{align*}
\]
Using the weights 0, 2, 1 and 1 yields
\[ \epsilon_4 + 18\gamma_3/4 + (9\gamma_2 - 10\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta \]
giving a coefficient 2.0 < 2.008 according to (41) and (43).
E) \( K(1) : r_6, r_2, r_3 \) and \( s = 0 \)
\[ \epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 6\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 6]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta. \]
Using the weights 1, 0, 0 and 2 yields
\[ \epsilon_4 + 9\gamma_3/3 + (4\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.76 < 2.008 according to (41) and (43).
F) \( K(1) : r_6, r_3, r_2 \) and \( s = 0 \)
\[ \epsilon_4 + 7\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 6\beta_1^{(4)} - 4\beta_1^{(3)} + [4, 6]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 6\beta_1^{(4)} + 4\beta_1^{(3)} - [4, 6]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [1, 2]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta. \]
Using the weights 1, 2, 0 and 0 yields
\[ \epsilon_4 + 15\gamma_3/3 + (8\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient 1.57 < 2.008 according to (41) and (43).
Thus the case where \( K(1) \) is represented by \( r_2, r_3, r_6 \) and \( s = 0 \) is excluded. and we have only one case left for \( K(1) \).
We now have a look at the main list. The transfer \( r_4 \) is excluded by the same argument as for \( K(1) \). If \( r_6 \) would occur in the main list then it would also occur in the list for \( K(1) \). But we already know it does not. So we are left with \( r_1, r_2, r_3, r_5 \) and \( s = 1 \) or \( s = 5 \). All other values for \( s \) would also work for \( K(1) \). But there we know \( s = 0 \).
Table 34 A. The main list $K(0)$ for $s = 1$ in interval $I_{10}$

<table>
<thead>
<tr>
<th>$r_5$</th>
<th>$r_3$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.90*</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.90*</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 5\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.36</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 5\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.36</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 6\gamma_3 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.62</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 6\gamma_3 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.62</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (6\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.14</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (6\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.14</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.85*</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.85*</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.36</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.36</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 12\gamma_3/3 + (8\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.94</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 12\gamma_3/3 + (8\gamma_2 - 9\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=1$</td>
<td>1.94</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 15\gamma_3/4 + (10\gamma_2 - 11\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.24</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 15\gamma_3/4 + (10\gamma_2 - 11\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=1$</td>
<td>2.24</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 34 B. The main list $K(0)$ for $s = 5$ in interval $I_{10}$
If one single line (not the last one) from the main list can be moved to the $K(1)$-list then the average inequality for the $K(1)$-list turns from
\[ \epsilon_4 + 8\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
into
\[ \epsilon_4 + 8\gamma_3/3 + (4\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient $1.98 < 2.008$ according to (41) and (43).
Many cases can now be excluded by using this argument.
1) Here the main list is represented by $r_2$ and $s = 1$. Thus the first line can be moved to the $K(1)$-list to give the additional contribution mentioned above.
2) Here the main list is represented by $r_1, r_2$ and $s = 1$.
Main list:
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2] \gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta. \]
List for $K(1)$:
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2,3] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2,3] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1,2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1,2] \gamma_1 \leq h + \delta. \]
Using the weights $0, 1, 0$ and $0, 1, 0$ yields
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient $1.97 < 2.008$ according to (41) and (43).

If we choose the other ordering for the $K(1)$-list we get
\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 & \leq h + \delta.
\end{align*}
\]

Averaging using the weights $0, 0, 1$ and $0, 1, 0$ again yields
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient $1.97 < 2.008$ according to (41) and (43).

3) Here the main list is represented by $r_2, r_3$ and $s = 1$. Here $r_2$ and $r_3$ follow each other and we get an additional contribution to the $K(1)$-list as described above.

4) Here the main list is represented by $r_1, r_2, r_3$ and $s = 1$. Again $r_2$ and $r_3$ follow each other and we get an additional contribution to the $K(1)$-list as described above.

5) Here the main list is represented by $r_2, r_5$ and $s = 1$. If $r_2$ comes first we get an additional contribution to the $K(1)$-list as described above. We now consider the case where $r_5$ comes first. We also have $\beta_1^{(4)} = 0$.

Main list:
\[
\begin{align*}
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(3)} + [3, 5]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]

List for $K(1)$:
\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]

Averaging using the weights $1, 1, 0$ and $0, 0, 1$ yields
\[ \epsilon_4 + 10\gamma_3/3 + (6\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
giving a coefficient $1.93 < 2.008$ according to (41) and (43).
If we choose the other ordering for the $K(1)$-list we get
\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} \leq h + \delta
\]
\[
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(3)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta.
\]
Averaging using the weights 0, 0, 1 and 0, 1, 0 again yields
\[
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient $1.97 < 2.008$ according to (41) and (43).

6) Here the main list is represented by $r_1, r_2, r_5$ and $s = 1$. We get two orderings for the main list and two for the $K(1)$-list.

Main list:
\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta
\]
\[
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_2^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.
\]
List for $K(1)$:
\[
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta.
\]
Averaging by using the weights 0, 0, 1, 0 and 1, 0, 0 yields
\[
\epsilon_4 + 7\gamma_3/2 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient $1.39 < 2.008$ according to (41) and (43).

If we choose the other ordering for the $K(1)$-list we get
\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta
\]
\[
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta.
\]
Averaging by using the weights 0, 0, 1 and 0, 1, 0 again yields
\[
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
giving a coefficient $1.97 < 2.008$ according to (41) and (43).

Now we consider the other ordering for the main list.
Main list:
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] \gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta. \]

List for \( K(1) \):
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 \leq h + \delta. \]

Averaging by using the weights 0, 1, 0, 0 and 0, 1, 0 yields
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.97 < 2.008 according to (41) and (43).

If we choose the other ordering for the \( K(1) \)-list we get
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} - [1, 2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 \leq h + \delta. \]

Averaging by using the weights 0, 0, 1, 0 and 0, 0, 1 again yields
\[ \epsilon_4 + 7\gamma_3/2 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
giving a coefficient 1.39 < 2.008 according to (41) and (43).

The main list is represented by \( r_2, r_3, r_5 \) and \( s = 1 \). If \( r_2 \) and \( r_3 \) follow each other in the main list they will have the same positions in the \( K(1) \)-list and we get an additional contribution and we are through. We get the same additional contribution if the main list starts with \( r_3 \) or \( r_2 \). But then no possible orderings are left and we are through.

The main list is represented by \( r_1, r_2, r_3, r_5 \) and \( s = 1 \). If \( r_2 \) and \( r_3 \) follow each other in the main list they will have the same positions in the \( K(1) \)-list and we get an additional contribution and we are through. But then there are only two orderings left, \( r_1, r_2, r_5, r_3 \) and \( s = 1 \) and \( r_1, r_3, r_5, r_2 \) and \( s = 1 \).

Main list:
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2] \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3] \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3] \gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta. \]
From the last line we know
\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta.\]
Averging this line and line 3 from the main list yields
\[\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\]
giving a coefficient $1.85 < 2.008$ according to (71).

Now we look at the second ordering of the main list:
\begin{align*}
\epsilon_4 &+ 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \\
\epsilon_4 &+ 4\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \\
\epsilon_4 &+ 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta.
\end{align*}

From the last line we know
\[\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta.
\]
Averging this line and line 3 from the main list yields
\[\epsilon_4 + 8\gamma_3/2 + (5\gamma_2 - 6\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta\]
giving a coefficient $1.85 < 2.008$ according to (71).

The following 4 cases have $s = 5$.

9) Here the main list is represented by $r_1, r_2$ and $s = 5$. If $r_2$ comes first we again get an additional contribution and we are through. If $r_2$ is in the second last line we get in the last line
\[\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5]\gamma_1 \leq h + \delta.\]
From this line we get
\[\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta.
\]
Combining this line with the first two lines from the $K(1)$-list we get
\[\epsilon_4 + 13\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.\]
giving a coefficient $2.0019 < 2.008$ according to (41) and (43).

10) Here the main list is represented by $r_1, r_2, r_3$ and $s = 5$. If $r_2$ and $r_3$ follow each other we again get an additional contribution and we are through. If $r_2$ or
Thus we finished interval $I_{10}$.

11) Here the main list is represented by $r_5, r_1, r_2$ and $s = 5$. If $r_2$ comes in the second last line we can argue as in case 9). Thus the only ordering we have to consider is $r_5, r_2, r_1$ and $s = 5$.

\[
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \gamma_1 - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta
\]

\[
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 5\beta_1^{(4)} + 3\beta_1^{(3)} - [3, 5] - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
\]

\[
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} \leq h + \delta
\]

\[
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + \beta_1^{(4)} - 5\beta_1^{(4)} - 3\beta_1^{(3)} + [3, 5]\gamma_1 \leq h + \delta
\]

List for $K(1)$:

\[
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta
\]

\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
\]

\[
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta
\]

Averaging by using the weights 0, 1, 0 , 0 and 1, 0, 0 yields

\[
\epsilon_4 + 7\gamma_3/2 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]

giving a coefficient $1.39 < 2.008$ according to (41) and (43).

If we choose the other ordering for the $K(1)$-list we get

\[
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta
\]

\[
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta
\]

\[
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta
\]

Averaging by using the weights 0, 0, 1, 0 and 0, 1, 0 again yields

\[
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]

giving a coefficient $1.97 < 2.008$ according to (41) and (43).

12) Here the main list is represented by $r_5, r_1, r_2, r_3$ and $s = 5$. If $r_2$ and $r_3$ follow each other we again get an additional contribution and we are through. If $r_2$ is in the second last line we can argue as in case 9). If $r_3$ is in the second last line, we may conclude from the last line that:

\[
\epsilon_4 + 6\gamma_3 + 4\gamma_2 - 5\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta
\]

Combining this line with the two first lines from the $K(1)$-list yields

\[
\epsilon_4 + 13\gamma_3/3 + (7\gamma_2 - 8\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]

giving a coefficient $2.0019 < 2.008$ according to (41) and (43).

Thus we finished interval $I_{10}$. 
12.12 Interval $I_{11}$

In this interval we start with the key number

$$M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1$$

and try to find a representation for $M(1)$ and all the numbers in the interval $K(1) = [M(1) - \gamma_1 + 1, M(1)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(1)$. Also here it turns out that only very few transfers are possible. None of the transfers from $D^*$ or $E$ can give optimal representations as long as $s > 0$ since these transfers produce second terms larger than $\gamma_2$.

In interval $I_{11}$ we have $\frac{4}{5}\gamma_2 < \beta_2^{(4)} < \frac{5}{6}\gamma_2$. Therefore we have

$$\gamma_2 > 6\beta_2^{(4)} - 4\gamma_2 > \beta_2^{(4)} > 2\beta_2^{(4)} - \gamma_2 > 3\beta_2^{(4)} - 2\gamma_2 > 4\beta_2^{(4)} - 3\gamma_2 > 5\beta_2^{(4)} - 4\gamma_2 > 0$$

and only $r_2, r_3, r_4$ and $r_5$ from $A$ may be used. We know that $s = 1$ is impossible and therefore $s = 0, 2, 3, 4$ or $s = 5$.

Now we assume that $(s_2, s_3, s_4) \in A$ is used. If $s_4 = 4$ or $s_4 = 5$ we get

$$\epsilon_4 + 5\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} \leq h + \delta.$$  

This contradicts (45) for $t = 4$ since $3\gamma_2 - 3\beta_2^{(4)} > \gamma_2/2$.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \leq 3$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$. If $s_4 \geq 4$ we get

$$\epsilon_4 + 5\gamma_3 + \beta_2^{(4)} \leq h + \delta.$$  

This contradicts (45) for $t = 4$ since $\beta_2^{(4)} > \gamma_2/2$.

Thus we are left with the possibilities $r_2$ and $r_3$ in addition to $s = 0, 2, 3$. All these possibilities are listed up in the table below. Only one case has a coefficient above 2.008.

**Table 35. K(1) for s > 0 in the main list in interval I_{11}**
Thus there is only one case left: \( r_1, r_2 \) and \( s = 0 \) for \( K(1) \). We now proceed to the main list. Later we will see that \( r_6 \) cannot occur there. We can not have \( s = 6 \) there neither. Since we postpone \( r_6 \) and \( s = 6 \) we only get the following possibilities for the main list: \( r_1, r_2, r_3 \) and \( s = 1 \). The cases \( s_4 = 4 \) and \( s_4 = 5 \) can be excluded by the same arguments as in the beginning of the section. If we had \( s = 2 \) or \( s = 3 \) these transfers would also be used in the final line for \( K(1) \) since they are possible there. But we have already seen that this is not the case. So we are left with \( s = 1 \) for the main list.

**Table 36. The main list for \( s > 0 \) in interval \( I_{11} \)**

<table>
<thead>
<tr>
<th>( r_2 )</th>
<th>( r_3 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coefficient</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + \gamma_3 + \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=0</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 3 \gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=0</td>
<td>1.42</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 4 \gamma_3 + 2 \gamma_2 - 2 \gamma_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=0</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 5 \gamma_3/2 + (2 \gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>1.37</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 7 \gamma_3/2 + (3 \gamma_2 - 3 \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=2</td>
<td>1.68</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 4 \gamma_3 + 2 \gamma_2 - 2 \gamma_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=3</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 4 \gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 3 \gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta )</td>
<td>s=2</td>
<td>1.42</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 7 \gamma_3/2 + (3 \gamma_2 - 3 \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=3</td>
<td>1.68</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 8 \gamma_3/3 + (3 \gamma_2 - 2 \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.25</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 7 \gamma_3/2 + (3 \gamma_2 - 3 \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=2</td>
<td>1.68</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 7 \gamma_3/2 + (3 \gamma_2 - 3 \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=3</td>
<td>1.68</td>
<td></td>
</tr>
</tbody>
</table>

In case A we combine the inequality \( \epsilon_4 + 2 \gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 \leq h + \delta \) with \( \epsilon_4 + 8 \gamma_3/3 + (3 \gamma_2 - 2 \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \) from \( K(1) \). We use the weight distribution 1 and 3 and get: \( \epsilon_4 + 10 \gamma_3/4 + (4 \gamma_2 - 3 \beta_2^{(4)})/4 + 2 \gamma_1/4 \leq h + \delta \) giving a coefficient 1.89 < 2.008 according to (71).

In case B we use \((s_2, 0, 1) \in C \) at the end of the main list. The transfer \( r_2 \) is used in the line before the last. The final line therefore reads:
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} + s_2\gamma_1 \leq h + \delta. \]

In the list for \(K(1)\) we get two orderings:

\[ \begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 & \leq h + \delta.
\end{align*} \]

Combining line two with the last line from the main list gives:

\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]

giving a coefficient \(1.92 < 2.008\) according to (71).

The other ordering for \(K(1)\) gives the following system:

\[ \begin{align*}
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 & \leq h + \delta.
\end{align*} \]

Here line two can be used to substitute the final line in the main list giving the following inequality there:

\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]

This is the same inequality as above and the case is solved.

In case C two of the lines from the main list fit in the list for \(M(1)\) and we get the additional contribution \(2(\gamma_2 - \beta_2^{(4)})\) giving

\[ \epsilon_4 + 8\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)} + 2(\gamma_2 - \beta_2^{(4)}))/2 + \gamma_1/2 = \epsilon_4 + 8\gamma_3/3 + (5\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]

yielding a coefficient \(1.92 < 2.008\) according to (41) and (43).

Case D splits into two subcases. If \(r_3\) stands in the second last line the two last lines read

\[ \begin{align*}
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - \beta_1^{(4)} + \gamma_1 & \leq h + \delta.
\end{align*} \]

Combining these two lines using the weights 2 and 1 yields:

\[ \epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]

giving a coefficient \(1.96 < 2.008\) according to (41) and (43).
If \( r_2 \) stands in the second last line the two lines in the middle of the main list read:

\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + 3\gamma_2 &- 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 &- 2\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta
\end{align*}
\]

Combining these two lines using the weights 1 and 2 gives:

\[
\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
\]

giving a coefficient 1.96 < 2.008 according to (41) and (43) and the case is closed.

Thus we only have to show that \( r_6 = 1 \) or \( s = 6 \) cannot occur in the main list. We try to find a representation for \( M(2) \) and all the numbers in the inverall \( K(2) = [M(2) - \gamma_1 + 1, M(2)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(2) \). Also here it turns out that only very few transfers are possible. None of the transfers from \( D^* \) or \( E \) can give optimal representations as long as \( s > 0 \) since these transfers produce second terms larger than \( \gamma_2 \).

Now we assume that \((s_2, s_3, s_4) \in A \) is used. \( s_4 = 6, 1 \) and \( s_4 = 2 \) are of course impossible. We know that \( r_3 \) is used in the list. For \( s_4 = 5 \) we get

\[
\epsilon_4 + 6\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} \leq h + \delta
\]

This contradicts (45) for \( t = 5 \) since \( 3\gamma_2 - 3\beta_2^{(4)} > \gamma_2/3 \).

Now we assume that \((s_2, s_3, s_4) \in B^* \) is used. If \( 1 < s_4 < 6 \) we have \( s_3 = s_4 \) and therefore we have

\[
\epsilon_4 + (1 + s_4)\gamma_3 + \gamma_2 + (s_4 - 2)(\gamma_2 - \beta_2^{(4)}) \leq h + \delta
\]

Here the second coefficient is \( \geq \gamma_2 \) which is impossible if \( s > 0 \). If \( s_4 = 6 \) we have

\[
\epsilon_4 + 7\gamma_3 + 4(\gamma_2 - \beta_2^{(4)}) \leq h + \delta
\]

This contradicts (45) for \( t = 6 \) since \( 4\gamma_2 - 4\beta_2^{(4)} > \gamma_2/6 \).

Thus we are left with the possibilities \( r_3, r_4 \) and \( q_1 \). We know that \( r_3 \) is used.

### Table 37. The list for \( K(2) \) for \( s > 0 \) in interval \( I_{11} \)

<table>
<thead>
<tr>
<th>( r_4 )</th>
<th>( r_3 )</th>
<th>( q_1 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 5\gamma_3/2 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>1.95</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 7\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.16</td>
<td>E</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>1.96</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 12\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>2.16</td>
<td>F</td>
</tr>
</tbody>
</table>
In case E we get an additional contribution from $K(1)$. The amount is $\gamma_2 - \beta_2^{(4)}$. Thus we get

$$\epsilon_4 + 7\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.$$  

Since we assume that $r_6$ or $s = 6$ is used in the main list we get

$$\epsilon_4 + 7\gamma_3 + 5\gamma_2 - 6\beta_2^{(4)} \leq h + \delta.$$  

We combine these inequalities using the weight distribution 3 and 1 and get

$$\epsilon_4 + 14\gamma_3/4 + (6\gamma_2 - 5\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta.$$  

giving a coefficient 1, 95 < 2.008 according to (41) and (43).

In case F we have to look at $M(3)$ and all the numbers in the interval $K(3) = [M(3) - \gamma_1 + 1, M(3)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(3)$. Also here it turns out that only very few transfers are possible. None of the transfers from $D^*$ or $E$ can give optimal representations as long as $s > 0$ since these transfers produce second terms larger than $\gamma_2$.

Now we assume that $(s_2, s_3, s_4) \in A$ is used. $s_4 = 6, 1, 2$ and $s_4 = 3$ are of course impossible. We know that $r_4$ is used in the list. For $s_4 = 5$ we get

$$\epsilon_4 + 6\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} \leq h + \delta.$$  

This contradicts (45) for $t = 5$ since $2\gamma_2 - 2\beta_2^{(4)} > \gamma_2/3$.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $2 < s_4 < 6$ we have $s_3 = s_4$ and therefore we have

$$\epsilon_4 + (1 + s_4)\gamma_3 + \gamma_2 + (s_4 - 3) + (\gamma_2 - \beta_2^{(4)}) \leq h + \delta.$$  

Here the second coefficient is $\geq \gamma_2$ which is impossible if $s > 0$. If $s_4 = 6$ we have

$$\epsilon_4 + 7\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} \leq h + \delta.$$  

This contradicts (45) for $t = 6$ since $3\gamma_2 - 3\beta_2^{(4)} > \gamma_2/6$.

Thus we are left with the possibilities $r_4, q_1$ and $q_2$. We know that $r_4$ and $q_1$ are used.

**Table 38. The list for $K(3)$ for $s > 0$ in the main list in interval $I_{11}$**

<table>
<thead>
<tr>
<th>$r_4$</th>
<th>$q_2$</th>
<th>$q_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 8\gamma_3/3 + (4\beta_2^{(4)} - 2\gamma_2)/3 + \gamma_1/3 \leq h + \delta$</td>
<td>$s=0$</td>
<td>2.41</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/4 + (5\beta_2^{(4)} - 2\gamma_2)/4 + \gamma_1/4 \leq h + \delta$</td>
<td>$s=0$</td>
<td>2.36</td>
</tr>
</tbody>
</table>
In the first case we get an additional contribution from $K(2)$. The amount is $2(\gamma_2 - \beta_2^{(4)})$. Thus we get

$$
\epsilon_4 + 8\gamma_3/3 + (4\beta_2^{(4)} - 2\gamma_2 + 2(\gamma_2 - \beta_2^{(4)}))/3 + \gamma_1/3 = \\
\epsilon_4 + 8\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta.
$$

giving a coefficient $1.92 < 2.008$ according to (41) and (43).

In the second case we get the same additional contribution and thus we have

$$
\epsilon_4 + 11\gamma_3/4 + (5\beta_2^{(4)} - 2\gamma_2 + 2(\gamma_2 - \beta_2^{(4)}))/4 + \gamma_1/4 = \\
\epsilon_4 + 11\gamma_3/4 + 3\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta.
$$

Since we assume that $r_6$ or $s = 6$ is used in the main list we get

$$
\epsilon_4 + 7\gamma_3 + 5\gamma_2 - 6\beta_2^{(4)} \leq h + \delta.
$$

We combine these inequalities using the weight distribution 4 and 1 and get

$$
\epsilon_4 + 18\gamma_3/5 + (5\gamma_2 - 3\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta.
$$

giving a coefficient $1.97 < 2.008$ according to (41) and (43) and interval $I_{11}$ is finished.

### 12.13 Interval $I_{12}$

We start considering the key number

$$
M(3) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (3\beta_2^{(4)} - 2\gamma_2 - 1)a_2 + \gamma_1 - 1
$$

and try to find a representation for $M(3)$ and all the numbers in the inverall $K(3) = [M(3) - \gamma_1 + 1, M(3)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(3)$. Also here it turns out that only very few transfers are possible. No transfers from $D^*$ and $E$ can be used since $s > 0$.

In interval $I_{12}$ we have $5\gamma_2 < \beta_2^{(4)} < \gamma_2$. Therefore we have

$$
\gamma_2 > \beta_2^{(4)} > 2\beta_2^{(4)} - \gamma_3 > 3\beta_2^{(4)} - 2\gamma_2 > 4\beta_2^{(4)} - 3\gamma_2 > 5\beta_2^{(4)} - 4\gamma_2 > 6\beta_2^{(4)} - 5\gamma_2 > 0
$$

and only $r_4, r_5$ and $r_6$ from $A$ may be used.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \geq 3$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$.

Thus we are left with the possibilities $r_4, r_5, r_6, q_1, q_2$ and $s = 0, 4, 5, 6$. Of course $0 < s \leq 3$ is impossible.

If $q_1$ is used we cannot have $s = 4, s = 5$ or $s = 6$ since the second coefficient is $3\beta_2^{(4)} - 2\gamma_2 - (\beta_2^{(4)} - \gamma_2) > 4\beta_2^{(4)} - 3\gamma_2$ and $q_1$ cannot produce the minimal
representation in $K(3)$. The same is true for $q_2$ where
\[ 3\beta_2^{(4)} - 2\gamma_2 - (2\beta_2^{(4)} - 2\gamma_2) > 4\beta_2^{(4)} - 3\gamma_2. \]
Now we compare a list where $s = 4$ stands at the end with corresponding list with the same transfers used before the last line where $s = 5$ or $s = 6$ stands at the end. If $s = 4$ produces the inequality
\[ \epsilon_4 + a\gamma_3/l + b\gamma_2/l + c\gamma_1/l \leq h + \delta \]
then $s = 5$ gives
\[ \epsilon_4 + (a + 1)\gamma_3/l + b\gamma_2/l + (\gamma_2 - \beta_2^{(4)})/l + c\gamma_1/l \leq h + \delta \]
giving a lower coefficient. Thus if the coefficient is below 2.008 for $s = 4$ it is even lower if $s = 5$. The same argument applies for $s = 6$. By the same reason we can see the following: If a list including $r_4$ but not $r_5$ gives a coefficient below 2.008 then the same list where $r_5$ substitutes $r_4$ gives an even lower coefficient and has to be excluded. Thus it is not necessary to calculate the inequalities and coefficients for $s = 5$ or $s = 6$ if the one for $s = 4$ already gives us a coefficient below 2.008. In these cases only the one for $s = 4$ is given. In the same way we can exclude a case where $q_2$ is used but not $q_1$ if the corresponding case where $q_2$ is replaced by $q_1$ already gives a coefficient below 2.008.
Table 39 A. $K(3)$ for $s > 0$ in the main list in interval $I_{12}$, Part I $r_6 = 0$

<table>
<thead>
<tr>
<th>$r_6$</th>
<th>$r_5$</th>
<th>$r_4$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>Average inequality</th>
<th>s</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} - 2\gamma_2 + \gamma_1 \leq h + \delta$</td>
<td>s=0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 4\gamma_3/2 + (4\beta_2^{(4)} - 2\gamma_2)/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 3\gamma_3/2 + (5\beta_2^{(4)} - 3\gamma_2)/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>2.28</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 6\gamma_3/3 + (6\beta_2^{(4)} - 3\gamma_2)/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 6\gamma_3/2 + (2\beta_2^{(4)} - \gamma_2)/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.85</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 10\gamma_3/2 + \gamma_2 - (2\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.98</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (3\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.85</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 8\gamma_3/3 + (4\beta_2^{(4)} - 2\gamma_2)/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.26</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 11\gamma_3/4 + (5\beta_2^{(4)} - 2\gamma_2)/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>2.24</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>see inequality system with $r_4$ in stead of $r_5$</td>
<td>s=0</td>
<td>1.85</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>see inequality system with $r_4$ in stead of $r_5$</td>
<td>s=0</td>
<td>1.80</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (3\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.85</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 12\gamma_3/4 + (4\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>1.95</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 12\gamma_3/3 + \gamma_2/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.77</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 15\gamma_3/4 + (\beta_2^{(4)} + \gamma_2)/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>1.82</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 14\gamma_3/4 + 2\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>2.06</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 17\gamma_3/5 + 3\beta_2^{(4)}/5 + \gamma_1/5 \leq h + \delta$</td>
<td>s=0</td>
<td>2.089</td>
</tr>
</tbody>
</table>
We now go through the cases where \( r_6 = 1 \). None of them has a coefficient over 2.008.

Table 39 B. \( K(3) \) for \( s > 0 \) in the main list in interval \( I_{12} \), Part II

\[
\begin{array}{cccccc}
\hline
r_6 & r_5 & r_4 & q_1 & q_2 & \text{Average inequality} & s & \text{Coeff.} \\
\hline
1 & 0 & 0 & 0 & 0 & \text{see inequality system with } r_5 \text{ in stead of } r_6 & s=0 & 1.35 \\
& & & & & \epsilon_4 + 12\gamma_3/2 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta & s \geq 4 & 1.65 \\
1 & 0 & 0 & 0 & 1 & \text{see inequality system with } r_5 \text{ in stead of } r_6 & s=0 & 1.85 \\
1 & 0 & 0 & 1 & 0 & \text{see inequality system with } r_5 \text{ in stead of } r_6 & s=0 & 1.85 \\
1 & 0 & 0 & 1 & 1 & \text{see inequality system with } r_5 \text{ in stead of } r_6 & s=0 & 1.95 \\
1 & 0 & 1 & 0 & 0 & \text{see inequality system with } r_5 \text{ in stead of } r_6 & s=0 & 1.78 \\
& & & & & \text{see inequality system with } r_5 \text{ in stead of } r_6 & s \geq 4 & 1.92^* \\
1 & 0 & 1 & 0 & 1 & \text{see inequality system with } r_5 \text{ in stead of } r_6 & s=0 & 1.82 \\
1 & 0 & 1 & 1 & 0 & \epsilon_4 + 15\gamma_3/4 + (\beta_2^{(4)} + \gamma_2)/4 + \gamma_1/4 \leq h + \delta & s=0 & 1.82 \\
1 & 0 & 1 & 1 & 1 & \epsilon_4 + 18\gamma_3/5 + (2\beta_2^{(4)} + \gamma_2)/5 + \gamma_1/5 \leq h + \delta & s=0 & 1.89 \\
1 & 1 & 0 & 0 & 0 & \text{see inequality system with } r_4 \text{ in stead of } r_5 & s=0 & 1.78 \\
& & & & & \text{see inequality system with } r_4 \text{ in stead of } r_5 & s \geq 4 & 1.92^* \\
1 & 1 & 0 & 0 & 1 & \text{see inequality system with } r_4 \text{ in stead of } r_5 & s=0 & 1.82 \\
1 & 1 & 0 & 1 & 0 & \text{see inequality system with } r_4 \text{ in stead of } r_5 & s=0 & 1.82 \\
1 & 1 & 0 & 1 & 1 & \text{see inequality system with } r_4 \text{ in stead of } r_5 & s=0 & 1.89 \\
1 & 1 & 1 & 0 & 0 & \epsilon_4 + 19\gamma_3/4 + (4\gamma_2 - 3\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta & s=0 & 1.89 \\
& & & & & \epsilon_4 + 18\gamma_3/3 + (6\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta & s \geq 4 & 1.92 \\
1 & 1 & 1 & 0 & 1 & \text{see inequality system with } q_1 \text{ in stead of } q_2 & s=0 & 1.89 \\
1 & 1 & 1 & 1 & 0 & \epsilon_4 + 21\gamma_3/5 + (3\gamma_2 - \beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta & s=0 & 1.90 \\
1 & 1 & 1 & 1 & 1 & \epsilon_4 + 24\gamma_3/6 + 3\gamma_2/6 + \gamma_1/6 \leq h + \delta & s=0 & 1.88 \\
\hline
\end{array}
\]

Thus we are left with six possibilities for \( K(3) \)

1) \( q_1 \) and \( s = 0 \)
2) \( q_1, q_2 \) and \( s = 0 \)
3) \( r_4, q_1 \) and \( s = 0 \)
4) \( r_4, q_1, q_2 \) and \( s = 0 \)
5) \( r_4, r_5, q_1 \) and \( s = 0 \)
6) \( r_4, r_5, q_1, q_2 \) and \( s = 0 \)

Important is that all cases have \( s = 0 \) and \( q_1 = 1 \): This means there is a minimal representation with a second coefficient

\[ 3\beta_2^{(4)} - 2\gamma_2 - \beta_1^{(2)} + \gamma_2 = 2\beta_2^{(4)} - \gamma_2. \]

Such a representation would allow the use of \( s \geq 3 \). Thus \( s \geq 3 \) is excluded from the whole problem also in all the other lists.

Now we consider the key number

\[ M(2) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (2\beta_2^{(4)} - \gamma_2 - 1)a_2 + \gamma_1 - 1 \]
and try to find a representation for $M(2)$ and all the numbers in the inverall $K(2) = [M(2) - \gamma_1 + 1, M(2)]$. First we check which transfers from the sets $A, B^*, C, D^*$ or $E$ can be used in the interval $K(2)$. Also here it turns out that only very few transfers are possible. No transfers from $D^*$ and $E$ can be used since $s > 0$.

The transfers $r_3, r_4$ and $r_5$ from $A$ may be used but $r_6$ is excluded. Otherwhise $r_6$ would have appeared in the list for $K(3)$, which we know it does not.

Now we assume that $(s_2, s_3, s_4) \in B^*$ is used. If $s_4 \geq 2$ the second term will be $\geq \gamma_2$ which is impossible since $s > 0$. We also know that $s = 1, 2$ is impossible. Otherwhise the second coefficient would be negative. So in this list we also have $s = 0$.

Thus we are left with the possibilities $r_3, r_4, r_5, q_1$ and $s = 0$.

We now divide the cases into two parts. The cases where $r_3$ appears and the ones where $r_3$ does not appear. We first look at the last cathegory. If $r_3$ does not appear in the list for $K(2)$ then the list contains exactly the same transfers as the list for $M(3)$. So we have to study the lists for $M(3)$ where the coefficient is above 2.008 and turn them into lists for $M(2)$. We have to study six cases.

1) $q_1$ and $s = 0$: Here

$$\epsilon_4 + 3\gamma_3/2 + (5\beta_2^{(4)} - 3\gamma_2)/2 + \gamma_1/2 \leq h + \delta.$$  

turns into

$$\epsilon_4 + 3\gamma_3/2 + (3\beta_2^{(4)} - \gamma_2)/2 + \gamma_1/2 \leq h + \delta.$$  

giving a coefficient 1.77 < 2.008 according to (41) and (43).

2) $q_1, q_2$ and $s = 0$: Here

$$\epsilon_4 + 6\gamma_3/3 + (6\beta_2^{(4)} - 3\gamma_2)/3 + \gamma_1/3 \leq h + \delta.$$  

turns into

$$\epsilon_4 + 6\gamma_3/3 + 3\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta.$$  

giving a coefficient 1.79 < 2.008 according to (41) and (43).

3) $r_4, q_1$ and $s = 0$: Here

$$\epsilon_4 + 8\gamma_3/3 + (4\beta_2^{(4)} - 2\gamma_2)/3 + \gamma_1/3 \leq h + \delta.$$  

turns into

$$\epsilon_4 + 8\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.$$  

giving a coefficient 1.78 < 2.008 according to (41) and (43).

4) $r_4, q_1, q_2$ and $s = 0$: Here

$$\epsilon_4 + 11\gamma_3/4 + (5\beta_2^{(4)} - 2\gamma_2)/4 + \gamma_1/24 \leq h + \delta.$$  

turns into

$$\epsilon_4 + 11\gamma_3/4 + (2\gamma_2 + \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta.$$
Thus there are five possibilities for $K(2)$. We now consider the possibility $r_3, r_4, r_5$ and $s = 0$ for $K(2)$. Then in $K(3)$ we have to use the transfers $r_4, r_5, q_1$ and $s = 0$ or $r_4, r_5, q_1, q_2$ and $s = 0$. In any case we get an additional contribution. The amount is $2(\gamma_2 - \beta_2^{(4)})$. Thus we get for the first case for $K(3)$

$$\epsilon_4 + 14\gamma_3/4 + (2\beta_2^{(4)} + 2(\gamma_2 - \beta_2^{(4)}))/4 + \gamma_1/4 = \epsilon_4 + 14\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta.$$ 

giving a coefficient $1.85 < 2.008$ according to (41) and (43).

For the second case for $K(3)$ we get

$$\epsilon_4 + 17\gamma_3/5 + (3\beta_2^{(4)} + 2(\gamma_2 - \beta_2^{(4)}))/5 + \gamma_1/5 = \epsilon_4 + 17\gamma_3/5 + (2\gamma_2 + \beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta.$$ 

giving a coefficient $1.93 < 2.008$ according to (41) and (43).

Thus the first cathegory. The second is found in the table below

**Table 40. $K(2)$ for $s > 0$ in the main list in interval $I_{12}$, $r_3 = 1$**

<table>
<thead>
<tr>
<th>$r_5$</th>
<th>$r_4$</th>
<th>$r_3$</th>
<th>$q_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 5\gamma_3/2 + \beta_2^{(4)}/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=0</td>
<td>1.88</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 7\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.19</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>2.13</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 12\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>2.16</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 11\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=0</td>
<td>1.93</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 13\gamma_3/4 + (3\gamma_2 - \beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>1.99</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 16\gamma_3/4 + (5\gamma_2 - 4\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta$</td>
<td>s=0</td>
<td>2.25</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 18\gamma_3/5 + (5\gamma_2 - 3\beta_2^{(4)})/5 + \gamma_1/5 \leq h + \delta$</td>
<td>s=0</td>
<td>2.22</td>
</tr>
</tbody>
</table>

Thus there are five possibilities for $K(2)$. We now consider the possibility $r_3, r_4, r_5$ and $s = 0$ for $K(2)$. Then in $K(3)$ we have to use the transfers $r_4, r_5, q_1$ and $s = 0$ or $r_4, r_5, q_1, q_2$ and $s = 0$. In any case we get an additional contribution. The amount is $2(\gamma_2 - \beta_2^{(4)})$. Thus we get for the first case for $K(3)$

$$\epsilon_4 + 14\gamma_3/4 + (2\beta_2^{(4)} + 2(\gamma_2 - \beta_2^{(4)}))/4 + \gamma_1/4 = \epsilon_4 + 14\gamma_3/4 + 2\gamma_2/4 + \gamma_1/4 \leq h + \delta.$$ 

giving a coefficient $1.85 < 2.008$ according to (41) and (43).
The same argument can be used if $K(2)$ is represented by the transfers $r_3, r_4, r_5, q_1$ and $s = 0$. Also here an additional contribution of $2(\gamma_2 - \beta_2^{(4)})$ is possible. Now there are still three cases left for $K(2)$.

Now we consider the case where $r_3, r_4$ and $s = 0$ are used in the list for $K(2)$ giving

$$\epsilon_4 + 10\gamma_3/3 + (2\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.$$ 

The two first possibilities for $K(3)$ do not apply here so we have to combine with the last four possibilities. The third possibility with one additional contribution gives

$$\epsilon_4 + 8\gamma_3/3 + (3\beta_2^{(4)} - \gamma_2)/3 + \gamma_1/3 \leq h + \delta$$

Combining the two inequalities using the weights 9 and 3 yields

$$\epsilon_4 + 38\gamma_3/12 + 5\gamma_2/12 + 4\gamma_1/12 \leq h + \delta.$$ 

giving a coefficient $1.98 < 2.008$ according to (41) and (43).

The fourth possibility for $K(3)$ with one additional contribution gives

$$\epsilon_4 + 11\gamma_3/4 + (4\beta_2^{(4)} - \gamma_2)/4 + \gamma_1/4 \leq h + \delta$$

Combining with the inequality for $K(2)$ using the weights 12 and 4 yields

$$\epsilon_4 + 51\gamma_3/16 + 7\gamma_2/16 + 5\gamma_1/16 \leq h + \delta.$$ 

giving a coefficient $1.98 < 2.008$ according to (41) and (43).

The fifth possibility for $K(3)$ with one additional contribution gives

$$\epsilon_4 + 14\gamma_3/4 + (\beta_2^{(4)} + \gamma_2)/4 + \gamma_1/4 \leq h + \delta$$

Combining with the inequality for $K(2)$ using the weights 3 and 4 yields

$$\epsilon_4 + 24\gamma_3/7 + 3\gamma_2/7 + 2\gamma_1/7 \leq h + \delta.$$ 

giving a coefficient $1.95 < 2.008$ according to (41) and (43).

The sixth possibility for $K(3)$ with one additional contribution gives

$$\epsilon_4 + 17\gamma_3/5 + (2\beta_2^{(4)} + \gamma_2)/5 + \gamma_1/5 \leq h + \delta$$

Combining with the inequality for $K(2)$ using the weights 6 and 5 yields

$$\epsilon_4 + 37\gamma_3/11 + 5\gamma_2/11 + 3\gamma_1/11 \leq h + \delta.$$ 

giving a coefficient $1.96 < 2.008$ according to (41) and (43).
Thus the possibility \( r_3, r_4 \) for \( K(2) \) is excluded and we have only two choices for \( K(2) \) left. Both contain \( q_1 \). This means that \( s = 2 \) is impossible in any list and \( s = 1 \) is the only possibility for the main list.

Now we consider the key number

\[
M(1) = (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\beta_2^{(4)} - 1)a_2 + \gamma_1 - 1
\]

and try to find a representation for \( M(1) \) and all the numbers in the interval \( K(1) = [M(1) - \gamma_1 + 1, M(1)] \). First we check which transfers from the sets \( A, B^*, C, D^* \) or \( E \) can be used in the interval \( K(1) \). Also here it turns out that only very few transfers are possible. No transfers from \( D^* \) and \( E \) can be used since \( s > 0 \).

\( r_2, r_3 \) and \( r_4 \) from \( A \) may be used but \( r_5 \) is excluded. Otherwise \( r_5 \) would have appeared in the list for \( K(2) \), which we know it does not. The same is true for \( r_6 \).

Now we assume that \( (s_2, s_3, s_4) \in B^* \) is used. The second term will be \( \geq \gamma_2 \) which is impossible since \( s > 0 \). We also know that \( s = 1 \) is impossible. Otherwise the second coefficient would be negative. So in this list we have \( s = 0 \).

and we are left with the possibilities \( r_2, r_3, r_4 \) and \( s = 0 \).

### Table 41. \( K(1) \) for \( s > 0 \) in the main list in interval \( I_{12} \)

<table>
<thead>
<tr>
<th>( r_2 )</th>
<th>( r_3 )</th>
<th>( r_4 )</th>
<th>Average inequality</th>
<th>( s )</th>
<th>Coeff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + \gamma_3/2 + \beta_2^{(4)} + \gamma_1 \leq 4 + \delta )</td>
<td>s=0</td>
<td>1.2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 2\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>1.33</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 5\gamma_3/2 + (2\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>1.6</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 10\gamma_3/3 + (5\gamma_2 - 4\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.13</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \epsilon_4 + 4\gamma_3/2 + \gamma_2/2 + \gamma_1/2 \leq h + \delta )</td>
<td>s=0</td>
<td>2.00</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( \epsilon_4 + 9\gamma_3/3 + (4\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.37</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( \epsilon_4 + 8\gamma_3/3 + (3\gamma_2 - 2\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta )</td>
<td>s=0</td>
<td>2.66</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \epsilon_4 + 13\gamma_3/4 + (6\gamma_2 - 5\beta_2^{(4)})/4 + \gamma_1/4 \leq h + \delta )</td>
<td>s=0</td>
<td>2.76</td>
</tr>
</tbody>
</table>

Thus we found four possibilities for \( K(1) \)

1) \( r_3, r_4 \) and \( s = 0 \)
2) \( r_2, r_4 \) and \( s = 0 \)
3) \( r_2, r_3 \) and \( s = 0 \)
4) \( r_2, r_3, r_4 \) and \( s = 0 \)

The first case implies that \( K(2) \) also is represented by \( r_3, r_4 \) and \( s = 0 \), a case we just have excluded.

The second choice for \( K(1) \) implies that \( K(2) \) has to be represented by \( r_3, r_4, q_1 \) and \( s = 0 \) a fact that again implies that \( K(3) \) is represented by \( r_4, q_1, q_2 \) and \( s = 0 \). If we combine the inequality \( \epsilon_4 + 9\gamma_3/3 + (4\gamma_2 - 3\beta_2^{(4)})/3 + \gamma_1/3 \leq 4 + \delta \)
for $K(1)$ with the one for $K(3)$ taking into account the possible additional contribution $\epsilon_4 + 11\gamma_3/4 + 3\beta_2^{(4)}/4 + \gamma_1/4 \leq 4 + \delta$ using the weights 3 and 4 we get

$$\epsilon_4 + 20\gamma_3/7 + 4\gamma_2/7 + 2\gamma_1/7 \leq h + \delta$$

giving a coefficient $1.93 < 2.008$ according to (41) and (43).

The fourth choice for $K(2)$ has to be represented by $r_3, r_4, q_1$ and $s = 0$ a fact that again implies that $K(3)$ is represented by $r_4, q_1, q_2$ and $s = 0$. If we combine the inequality

$$\epsilon_4 + 12\gamma_3/4 + (4\gamma_2 - 2\beta_2^{(4)})/4 + \gamma_1/4 \leq 4 + \delta$$

does not give a positive gain. In (41) and (43).

Thus there is only one choice left for $K(1)$, namely $r_2, r_3$ and $s = 0$.

We now turn to the main list, where we know that only $r_1, r_2, r_3$ and $s = 1$ can occur. Since $r_1, r_5, r_6$ did not occur in the list for $K(1)$ they cannot occur in the main list either.

\[\text{Table 42. } K(0) \text{ for } s > 0 \text{ in the main list in interval } I_{12}\]

<table>
<thead>
<tr>
<th>$r_3$</th>
<th>$r_2$</th>
<th>$r_1$</th>
<th>Average inequality</th>
<th>$s$</th>
<th>Coeff.</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)}/4 + \gamma_1 \leq h + \delta$</td>
<td>s=1</td>
<td>2.34</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)}/4 + \gamma_1 \leq h + \delta$</td>
<td>s=1</td>
<td>2.34</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 5\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>3.30</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 5\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>3.96</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\epsilon_4 + 6\gamma_3/2 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>3.30</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\epsilon_4 + 6\gamma_3/2 + (4\gamma_2 - 4\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta$</td>
<td>s=1</td>
<td>3.30</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (6\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=1</td>
<td>3.84</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\epsilon_4 + 9\gamma_3/3 + (6\gamma_2 - 6\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta$</td>
<td>s=1</td>
<td>3.84</td>
<td>8</td>
</tr>
</tbody>
</table>

The first two cases imply that the main list has length 1 with a constant term $\gamma_1 - 1$. This means that the transfer $(0, 1, 0)$ does not give a positive gain. In this cases we have

$$\gamma_3 + \gamma_2 + \gamma_1 \leq h + \delta.$$

Combining this inequality with $\epsilon_4 + 7\gamma_3/3 + 2\beta_2^{(4)}/3 + \gamma_1/3 \leq h + \delta$ for $K(2)$ using the weights 3 and 10 yields

$$\epsilon_4 + 79\gamma_3/30 + 77\gamma_2/90 + 19\gamma_1/30 \leq 1.3h + \delta$$

giving a coefficient $2.001 < 2.008$ according to (42).
If $K(3)$ is represented by $r_4, q_1, q_2$ and $s = 0$ we get
\[ \epsilon_4 + 11\gamma_3/4 + (5\beta_2^{(4)} - 2\gamma_2)/4 + \gamma_1/4 \leq h + \delta. \]
Combining this inequality with $\gamma_3 + \gamma_2 + \gamma_1 \leq h + \delta$ using the weights 2 and 1 yields
\[ \epsilon_4 + 13\gamma_3/4 + 25\gamma_2/24 + 3\gamma_1/4 \leq 1.5h + \delta \]
giving a coefficient 1.99 < 2.008 according to (42)).

In case 4) the main list consists of $r_2$ and $s = 1$. Here $\kappa_l = \kappa_2 = \beta_1^{(4)} = 0$ and the list reads
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \beta_1^{(3)} \leq h + \delta. \]

For $K(1)$ we get
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(3)} - 2\beta_1^{(4)} + \gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(3)} - \gamma_1 \leq h + \delta. \]

Combining these two inequality systems using the weights 0, 1 and 0, 1, 0 gives:
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta. \]

Lateron we will see that this is enough to exclude the case.

In case 4) the main list consists of $r_1, r_2$ and $s = 1$.

\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(3)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(4)} - [1, 2]\gamma_1 - \beta_1^{(4)} + \gamma_1 \leq h + \delta. \]

For $K(1)$ we get two orderings.
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 \leq h + \delta. \]

Combining these two inequality systems using the weights 0, 0, 1 and 0, 1, 0 again gives:
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta. \]

The other ordering gives
\[ \epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 \leq h + \delta \]
\[ \epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 \leq h + \delta. \]
Combining these two inequality systems using the weights 0, 1, 0 and 0, 1, 0 again gives:
\[
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.
\]
5 and 6) Both the fifth and sixth possibility imply
\[
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta
\]
and we may turn to the seventh case. 7) In this case the main list consists of \(r_3, r_2\) and \(s = 1\). Here we have \(\beta_1^{(4)} = 0\) as we showed in the beginning of this chapter. Therefore \([1, 2] = 0\) The list for \(K(1)\) reads (two orderings):
\[
\begin{align*}
\epsilon_4 + 3\gamma_3 + \gamma_2 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 & \leq h + \delta.
\end{align*}
\]
We therefore have \([2, 3] = 1\). For \(K(2)\) we have
\[
\begin{align*}
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + \gamma_2 + \beta_1^{(3)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + 2\beta_1^{(3)} - [2, 3]\gamma_1 & \leq h + \delta.
\end{align*}
\]
Using the weights 0, 1, 1 and 1, 0, 0 yields
\[
\epsilon_4 + 7\gamma_3/3 + 2\gamma_2/3 + \gamma_1/3 \leq h + \delta
\]
giving a coefficient 1.88 < 2.008 according to (41) and (43).
The other ordering gives the following lists for \(K(1)\) and \(K(2)\):
\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]
We therefore have \([2, 3] = 0\). For \(K(2)\) we have
\[
\begin{align*}
\epsilon_4 + 4\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \beta_1^{(3)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} - \gamma_2 + \beta_1^{(3)} & \leq h + \delta.
\end{align*}
\]
Using the weights 1, 0, 1 and 0, 1, 0 yields again
\[
\epsilon_4 + 7\gamma_3/3 + 2\gamma_2/3 + \gamma_1/3 \leq h + \delta
\]
The weight distribution 0, 0, 2 and 1 yields $\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta$

In case 8) we have the following two alternatives for the main list:

$$
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \beta_1^{(4)} - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - \beta_1^{(4)} + \gamma_1 & \leq h + \delta
\end{align*}
$$

The weight distribution 0, 0, 2 and 1 yields

$$
\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
$$

Lateron we show that this inequality is enough to exclude the case. The second alternative for the main list reads:

$$
\begin{align*}
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + 4\gamma_3 + 3\gamma_2 - 3\beta_2^{(4)} + \beta_1^{(4)} - 3\beta_1^{(4)} - 2\beta_1^{(3)} + [2, 3]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 3\gamma_3 + 2\gamma_2 - 2\beta_2^{(4)} + 3\beta_1^{(4)} + 2\beta_1^{(3)} - [2, 3]\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} + [1, 2]\gamma_1 & \leq h + \delta \\
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - [1, 2]\gamma_1 - \beta_1^{(4)} + \gamma_1 & \leq h + \delta
\end{align*}
$$

The weight distribution 0, 1, 2 and 0 again yields

$$
\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta.
$$

Lateron we show that this inequality is enough to exclude the case. Now we are left with two cases for the main list

$$
\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$

and

$$
\epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - 3\beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta.
$$

In addition we have two cases for $K(2)$, namely $r_3, q_1$ and $s = 0$, giving

$$
\epsilon_4 + 7\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta,
$$

where we took an additional contribution into account, and $r_3, r_4, q_1$ and $s = 0$. The last one implies that $K(3)$ is represented by $r_4, q_1, q_2$ and $s = 0$, giving

$$
\epsilon_4 + 11\gamma_3/4 + 3\beta_2^{(4)}/4 + \gamma_1/4 \leq h + \delta.
$$

We now combine these cases and get four combinations.

We combine

$$
\epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - 7\beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta
$$
with
\[ \epsilon_4 + 7\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta, \]
using the weights 3 and 21. Then we get
\[ \epsilon_4 + 59\gamma_3/24 + 14\gamma_2/24 + 8\gamma_1/24 \leq h + \delta \]
giving a coefficient 2.006 < 2.008 according to (41) and (43).

Now we combine
\[ \epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta \]
with
\[ \epsilon_4 + 10\gamma_3/3 + (7\gamma_2 - \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta, \]
using the weights 9 and 28. Then we get
\[ \epsilon_4 + 107\gamma_3/37 + 21\gamma_2/37 + 10\gamma_1/37 \leq h + \delta \]
giving a coefficient 1.98 < 2.008 according to (41) and (43).

The next step is to combine
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
with
\[ \epsilon_4 + 7\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta, \]
Using the weights 2 and 9 we get
\[ \epsilon_4 + 27\gamma_3/11 + 6\gamma_2/11 + 4\gamma_1/11 \leq h + \delta \]
giving a coefficient 1.99 < 2.008 according to (41) and (43).

The last step is to combine
\[ \epsilon_4 + 6\gamma_3/2 + (3\gamma_2 - \beta_2^{(4)})/2 + \gamma_1/2 \leq h + \delta \]
with
\[ \epsilon_4 + 7\gamma_3/3 + (\gamma_2 + \beta_2^{(4)})/3 + \gamma_1/3 \leq h + \delta. \]
Using the weights 2 and 4 we get
\[ \epsilon_4 + 17\gamma_3/6 + 3\gamma_2/6 + 2\gamma_1/6 \leq h + \delta \]
giving a coefficient 1.96 < 2.008 according to (41) and (43).
Thus we finished interval $I_{12}$. 
13 The computer programme

In this section we present the computer programme. The programme covers the case \( s = 0 \) in the main list. The case \( s > 0 \) is already settled in the previous chapter. Three different programmes cover different stages in the process. We start with the first programme where sets of transfers are produced that have average lists with coefficients over 2.008. We choose the interval number \( p \) and the programme runs through the possible values of \( r_i, q_i \) and \( d_i \). These values are 0 or 1, according to whether the transfer is used or not used in the list. The programme then computes the average inequality and the corresponding coefficient according to (41) and (43). If the coefficient is below 2.008 the programme discards the set of transfers and goes to the next combination. If the coefficient is above 2.008 the set of transfers is stored on a file.

```python
import math;
```  
This programme produces main lists. You have to choose the interval number \( p \). The programme runs through the actual values for \( r_i, q_i \) and \( d_i \). We have \( s=0 \). The results are filed on a file. The name of which is given in the last line of the programme. Besides \( p \) no other choices are necessary.

In the following lines the variables used in the programme are defined.

```python
w=[None] * 14; g=[None] * 7; re=[None] * 17; rr=[None] * 17;
alin=[None] * 17; blin=[None] * 17; betalin=[None] * 17; t=[None] * 7;
SS=''
```

In the following lines the procedure that computes the coefficient from an inequality

\[
\epsilon_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta
\]

according to (43) is presented.
def kf ( a, b, c ):
    r = (3*(b+c)-2)/4; t = (b+c-4*b*c)/2; xx = math.sqrt(r*r+t)-r;
    if (xx >0):
        x=xx;
    else:
        x=0;
    return math.pow(1+ x,4)/(a*(b+x)*(c+x));

Since $\beta_2^{(4)}$ enters many of the inequalities but we need an expression containing $\gamma_2$ we here present a procedure that replaces $\beta_2^{(4)}$ with the upper or the lower bound according to the sign of the prefactor antbet.

def Betval (antbet) :
    if (antbet>0) :
        return w[p];
    else :
        return w[p+1];

Here a procedure is presented that picks two transfers from a list and combines them using different weights like in (52). A coefficient is computed. If the coefficient is below 2.008 this can help to reduce the number of cases to consider.

def kfs(rr):
    M=10;j=1;opts=10;opt=10;
    while ( j<17 ):
        if (rr[j]==1):
            i=1;
        while ( i<=j-1 ):
            if (rr[i]==1):
                jj=1;
            while ( jj<=M ):
                aa=(jj*alin[j]+(M-jj)*alin[i])/M;bb=(jj*blin[j]+(M-jj)*blin[i])/M;
                antbet=(jj*betalin[j]+(M-jj)*betalin[i])/M;bb=bb+antbet*Betval(antbet);
                opt=kf(aa,bb,0);
                if (opt<opts):
                    opts=opt;
                jj+=1;
            i+=1;
        j+=1;
    return opts;

Here the $s_3$ values are computed according to the choice of $s_4$ for transfers from $A$. 
The different transfers that may occur in a list are ordered. First we have the transfers \( r_1, r_2, r_3, r_4, r_5, r_6 \in A \). Then we have \( q_1, q_2, q_3, q_4, q_5, q_6 \in B^* \) and in the end we have \( d_1, d_2, d_3 \in D^* \) and finally we get \( q \in E \).

In the following lines the ends of the loops are computed. Usually the ends are \( re[i] = 1 \). But when we in a given interval \( p \) know that the corresponding transfer is impossible, then we put \( re[i] = 0 \). The output file is prepared, the first line containing a number of zeroes.
Here the real programme starts. We go through all possibilities for the $r_i, q_i$ and $d_i$. Then we collect all this information in a vector $rr$. Then the average inequality (47) is computed. Only lists of length $L \geq 1$ are considered. Otherwise the coefficient is 1. We compute the coefficient belonging to the average inequality and if it is above 2.008 we store the vector $rr$ recording the used transfers and the actual coefficient in a file and print it on the screen.

```python
print(" Ends of the loops ",re[1],re[2],re[3],re[4],re[5],re[6],re[7],re[8],re[9],
      re[10],re[11],re[12],re[13],re[14],re[15],re[16]);
SS+=str(0)+" ";SS+=str(0)+" ";SS+=str(0)+"\n";
```

```python
r6=0;
while (r6<=re[6]):
r5=0;
while (r5<=re[5]):
r4=0;
while (r4<=re[4]):
r3=0;
while (r3<=re[3]):
r2=0;
while (r2<=re[2]):
r1=0;
while (r1<=re[1]):
  q2=0;
  while (q2<=re[8]):
    q1=0;
    while (q1<=re[7]):
      d3=0;
      while (d3<=re[15]):
        d2=0;
        while (d2<=re[14]):
          d1=0;
          while (d1<=re[13] ):
            L=1;j=1;
            while (j<17):
              L+=rr[j];j+=1;
              optund=10;
              if (L>1) :
                optund=kfs(rr);
                ss4=0;ss3=d1+2*d2+3*d3;j=1;
              while (j<=6):
```
ss4+=j*(rr[j]+rr[j+6]);
ss3+=((rr[j]+rr[j+6])*g[j]+rr[j+6]);
j+=1;
a=1+ss4/L;b=ss3/L+1-(ss4/L)*Betval(-ss4);
c=1/L;opt1=kf(a,b,c);
if (opt1>bound) and (optund>bound):
counter1+=1;
print(counter1,"",rr,math.floor(10000*opt1));

tt=0;rr[0]=0;
while (tt<16):
    SS+=str(rr[tt])+
    tt+=1;
    SS+=str(math.floor(10000*opt1))\n
    if (opt1>total):
        total=opt1;
        d1+=1;
d2+=1;
d3+=1;
    q1+=1;
    q2+=1;
r1+=1;
r2+=1;
r3+=1;
r4+=1;
r5+=1;
r6+=1;
print("Max coefficient ",total," Number of cases ",counter1);
gg=open("res11.txt","w");gg.write(SS);gg.close();

In the present version of the programme p = 11 was chosen. If p ≠ 11 the name
of the output file has to be changed, f.ex.

"res.7.txt"

The second programme starts with a list of transfers where the coefficient is
above 2.008 from the file stored by the first programme. Then it goes through
all possible orderings of the transfers. Once an ordering is chosen we can
construct an inequality system. Usually these systems contain few lines, the
maximum number being 8 lines. A special procedure now chooses weights for
these lines and computes a coefficient for the h-range. Experiences in the first
test runs showed us that a total weight (sum of all weights) up to 12 was
sufficient for most of the systems. If this optimization procedure produces a
coefficient below 2.008 the corresponding ordering of the transfers is discarded,
otherwise the inequality system is stored in a file.

This programme needs the interval number, "p". In addition the programme
needs the number of cases "anttil" stored on the corresponding file.
Then the programme reads the file and produces an inequality system.
A coefficient is computed by running through suitable weights
for the lines of the inequality system.
If the coefficient is above 2.008 the inequality system is stored in a file.
In order to reduce the number of cases we choose values for beta13
and beta14 from a grid. These then give us the actual ordering and the
values of the s_2 in front of gamma_1.
Usually very few orderings survive compared to the number of
possibilities which arise combinatorically.

Here the variables used in the programme are defined.

```python
import math;
w=[None] * 14;g=[None] * 7;re=[None] * 17;rr=[None] * 17;
A=[None] * 22;xend=[None] * 9;pos=[None] * 11;ord=[None] * 11;
s3f=[None] * 51;s4f=[None] * 51;s2f=[None] * 51;kappaf=[None] * 51;
s3=[None] * 51;s4=[None] * 51;s2=[None] * 51;
f=[None] * 51;e=[None] * 51;cv=[None] * 51;av=[None] * 51;
bv=[None] * 51;betav=[None] * 51;Erf=[None] * 31;huskx=[None] * 51;
y=[None] * 9;
SS='';
plass=L=x1=x2=x3=x4=x5=x6=x7=x8=tell=i=antall=beta=Nev=k=j=0;
repet=n=h=p1=p2=sum=TELL0PP=t1=b13=b14=ii=teller=E=F=zahltotal=0;
bloknr=bl=behag=bhg1=bhg2=jj=LLL=huskteller=til=ihuisk=husk=counter=0;
zzz=0.0;s2[0]=-1;s3[0]=0;s4[0]=0;huska=husk=b=0.0;
a=b=c=opt=kappaf=total=0.0;eps=0.0000001;opt=10;total=1;opt1=10;
bound=2.008;
Er=[None] * 61;
j=0;
while (j<61):
    Er[j]=[None] * 31;j+=1;
```
Here we choose the interval number $p$ and the maximal total weight $N$. Usually $N = 12$ is sufficient. The name of the input file and the output file is computed automatically. The input file is read and the inequality systems are shown on the screen.

```python
p=3;anttil=15;m=0;M=0;N=12;filename="data"+str(p)+"niv0.txt";

ff=open('res'+str(p)+".txt");
A = [];
for line in ff: # read rest of lines
    A.append([int(x) for x in line.split()]);
ff.close();
```

In the following lines the procedure that computes the coefficient from an inequality

$$\epsilon_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta$$

according to (43) is presented. The procedure also contains an option for "pleasant" bases $A_3$. In this case we may assume

$$\gamma_3 + \gamma_2 + \gamma_1 \leq h + \delta,$$

which is a strong tool and the corresponding inequality systems often have small coefficients.

```python
def kfbehag (a,b,c,behag):
    r = (3*(b+c)-2)/4;t = (b+c-4*b*c)/2;
    xx = math.sqrt(r*r+t)-r;
    if (xx >0):
        x=xx;
    else:
        x=0;
    opt=math.pow(1+ x,4)/(a*(b+x)*(c+x));
    if behag==1 :
        h=1;
        while (h<=50) :
            x1=h/50;mid=(1+x1)*(1+x1)*(1+x1)*(1+x1)/((a+x1)*(b+x1)*(c+x1));
            if mid<opt :
                opt=mid;
                h+=1;
        return opt;
```
Since $\beta_2^{(4)}$ enters many of the inequalities but we need an expression containing $\gamma_2$ we here present a procedure that replaces $\beta_2^{(4)}$ with the upper or the lower bound according to the sign of the prefactor $\text{antbet}$.

```python
def Betval(antbet):
    if (antbet>0):
        return w[p];
    else:
        return w[p+1];
```

Here the $s_3$ values are computed according to the choice of $s_4$ for transfers from $A$.

```python
g[0]=-1;i=1;
while( i<7 ):
    g[i]=math.floor(i*w[p]+eps);i+=1;
```

Here the total input file is read and the content is displayed on the screen.

```python
i=0;
while (i<len(A)) :
    j=0;
    while (j<len(A[i])) :
        print (A[i][j], end=" ", flush=True);
        j+=1;
    print ("");i+=1;
```

Here the output file is prepared and the actual line of the input file is printed again.

```python
SS+=str(0)+' '+ss+=str(0)+' '+ss+=str(0)+'
;z1=1;
while (z1<anttil) :
    print("--------------------------");
    print(A[z1][1],A[z1][2],A[z1][3],A[z1][4],A[z1][5],A[z1][6],A[z1][7],
    A[z1][8],A[z1][9],A[z1][10],A[z1][11],A[z1][12],A[z1][13],A[z1][14],
    A[z1][15],A[z1][16],A[z1][17]);
```

Here the transfers get their preliminary ordering.

```python
zahltotal=0;teller=0;opt=10;tell=0;
i=1;tell=0;
while(i<=16):
```
if (A[z1][i]==1) :
    tell=tell+1;
    if (i<7):
        s4f[tell]=i; s3f[tell]=g[i];
    if (i>6) and (i<13):
        s4f[tell]=i-6; s3f[tell]=g[i-6]+1;
    if (i>12) and (i<16):
        s4f[tell]=0; s3f[tell]=i-12;
    if (i==16):
        s4f[tell]=1; s3f[tell]=2; i+=1;

L=tell+1;
print(" Length ",L);
j=L;
while (j<=10):
    s4f[j]=0; s3f[j]=0; j+=1;

Here we go through possible values of $\beta^{(4)}_1/\gamma_1$ and $\beta^{(3)}_1/\gamma_1$ in order to determine the ordering of the transfers and the values of $s_2$ for the transfers.

t1=0; p1=29; p2=31; b14=1;
while (b14<p1):
    b13=1;
    while (b13<p2):

Here we compute the actual values of $s_2$ for the transfers.

    j=1;
    while (j<=10):
        s2f[j]=math.floor(s3f[j]*b13/p2+s4f[j]*b14/p1);
        j+=1;

We now look at the transfer that has got the number $j$ in the preliminary ordering. We compare the reduction of the constant term with all other reductions of the constant terms from the list. Every time we find a transfer with a greater reduction of the constant term our transfer is moved one position further. When we have gone through all transfers we have found the correct position of the transfer $j$ under consideration. Thus we find the position of all transfers of the list. Now the reduction of the constant terms are ordered in decreasing order thus the constant terms of all lines are non-negative. Since every reduction is below $\gamma_1$ by definition, we know that the constant terms are between 0 and $\gamma_1$. 
j=1;  
while (j<L) :  
plass=1;  
i=1;  
while (i<L):  
  if p1*(s3f[j]-s3f[i])*b13+p2*(s4f[j]-s4f[i])*b14-(s2f[j]-s2f[i])*p1*p2<0 :  
    #kappaf[i]-s2f[i]>kappaf[j]-s2f[j] :  
    plass=plass+1;  
    i+=1;  
  ord[plass]=j;  
  j+=1;  
j=L;  
while (j<=10):  
  ord[j]=j;  
  j+=1;  

Here the transfers are listed according to the actual ordering.

j=1;  
while (j<=10) :  
h=3*(j-1);k=ord[j];Erf[h+1]=s4f[k];Erf[h+2]=s3f[k];Erf[h+3]=s2f[k];  
j+=1;  

Here we determine whether an ordering of the transfers has occurred before or not. If it is new it is stored and displayed on the screen.

repet=0;ii=1;  
while (ii<=t1):  
  sum=0; j=1;  
  while (j<=30):  
    if (Er[ii][j]==Erf[j]):  
      sum=sum+1;  
      j+=1;  
    if (sum==30) :  
      repet=1;  
      ii+=1;  

if ((repet==0) or (t1==0)) :  
  j=1;  
  while (j<=10):  
    h=3*(j-1);s4[j]=Erf[h+1];s3[j]=Erf[h+2];s2[j]=Erf[h+3];  
    cv[j]=s2[j]-s2[j-1];e[j]=s4[j-1]-s4[j];f[j]=s3[j-1]-s3[j];
Here we start a loop where we go through all of the possible orderings of the transfers. First they are listed. Then we determine whether the partial basis $A_3$ is pleasant or not. If it is pleasant it is much easier to exclude the case.

n=1;
while (n<=TELLOPP):  
    print("","Er[n][4],Er[n][5],Er[n][6],"")
    print("","Er[n][7],Er[n][8],Er[n][9],"")
    print("","Er[n][10],Er[n][11],Er[n][12],"")
    print("","Er[n][13],Er[n][14],Er[n][15],"")
    print("","Er[n][16],Er[n][17],Er[n][18],"")
    print("","Er[n][19],Er[n][20],Er[n][21],"")
    print("","Er[n][22],Er[n][23],Er[n][24],"");
    j=1;
    while (j<=L):
        h=3*(j-1);s4[j]=Er[n][h+1];s3[j]=Er[n][h+2];s2[j]=Er[n][h+3];
        av[j]=1+s4[j];bv[j]=-g[m]+s3[j];cv[j]=-s2[j-1]+s2[j];
        betav[j]=m-s4[j];e[j]=s4[j-1]-s4[j];
        f[j]=s3[j-1]-s3[j];huskx[j]=0;
        j+=1;
        j=L+1;
    while (j<=10):
        j+=1;
        behag=0;bhg1=0;bhg2=0;
        j=1;
        while (j<=L):
            if ((e[j]==1) and (f[j]==1) and (cv[j]==0)):
                behag=1; bhg1=1; bhg2=1;
            i=1;
            while (i<=L) :
if (e[j]==e[i]) and (f[j]>f[i]) and (cv[j]==cv[i]) :
    behag=1; bbg2=1; ihusk=i; jhusk=j;
    i+=1;
    j+=1;

Here we start the optimization process by choosing weights $x_1, x_2, \ldots, x_8$. The total weight is always $\leq 12$. This value showed up to be sufficient during the first experiments. The number $E$ gives us the accumulated amount of the $\beta_1^{(4)}$ whilst $F$ stands for the accumulated amount of $\beta_1^{(3)}$. If $F = E = 0$ the weight combination makes $\beta_1^{(4)}$ and $\beta_1^{(3)}$ disappear and we have an inequality of the kind

$$\epsilon_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta$$

when we consider the bounds for $\beta_2^{(4)}$. For this situation (43) gives us a way to compute the coefficient. When $c \leq 0$ and either $E = 0$ and $F = -1$ or $E = -1$ and $F = 0$ the "constant term" in the weighted average inequality is negative, which is impossible. In this case we put the coefficient 2. When $c > 0$ and either $E = 0$ and $F = -1$ or $E = -1$ and $F = 0$ the "constant term" in the weighted average inequality is $\geq (c-1)\gamma_1$ and we may use $c-1$ as the number of $\gamma_1$ in the formula when computing the coefficient. If all coefficients for all distributions of weights are above 2.008 then the inequality system is stored on a file and the programme proceeds to the next case on the input file.

teller=teller+1; total=10;
j=1;
while (j<=8):
    if j<=L:
        xend[j]=1;
    if j>L:
        xend[j]=0;
    j+=1;
    x1=0;
while (x1<=N)and (total>bound):
    x2=0;
    while (x2<=xend[2]*(N-x1)) and (total>bound):
        x3=0;
        while (x3<=xend[3]*(N-x1-x2)) and (total>bound):
            x4=0;
            while (x4<=xend[4]*(N-x1-x2-x3)) and (total>bound):
                x5=0;
                while (x5<=xend[5]*(N-x1-x2-x3-x4)) and (total>bound):
                    x6=0;
                    while (x6<=xend[6]*(N-x1-x2-x3-x4-x5)) and (total>bound):
x7=0;
while (x7<=xend[7]*(N-x1-x2-x3-x4-x5-x6)) and (total>bound):
x8=0;
while (x8<=xend[8]*(N-x1-x2-x3-x4-x5-x6-x7)) and (total>bound):
Nev=0;E=0;F=0;a=0;antall=0;b=0;c=0;
i=1;
while (i<=8):
  Nev=Nev+y[i];E=E+e[i]*y[i];F=F+f[i]*y[i];
a=a+av[i]*y[i];antall=antall+betav[i]*y[i];
b=b+bv[i]*y[i];c=c+cv[i]*y[i];
i+=1;
opt=10;
T1=(Nev>0);T2=((F==0) and (E==0));T3=((c>0) and (((F==0) and (E==-1)) or ((F==-1) and (E==0))));
T4=((c<=0) and (((F==0) and (E==-1)) or ((F==-1) and (E==0))));
if T1 and (T2 or T3 or T4):
  if T3:
    c=c-1;
opt=kfbehag(a/Nev,(b+antall*Betval(antall))/Nev,c/Nev,behag);
  if T4:
    opt=2;
  if (opt<total):
    total=opt;
i=1;
  while (i<=8):
    huskx[i]=y[i];
i+=1;
x8+=1;
x7+=1;
x6+=1;
x5+=1;
x4+=1;
x3+=1;
x2+=1;
x1+=1;
zahltotal=zahltotal+1;
print(z1,"Coef ",total,"Unequality system below.","pleasant",behag);
j=1;
if total >2.008:
  counter=counter+1;SS+=str(0)+" ";print("Anzahl bisher ",counter);
  SS+=str(A[z1][1])+" "; SS+=str(A[z1][2])+" ";SS+=str(A[z1][3])+" ";
Now the third programme reads an inequality system from an input file generated by the second programme, finds the transfer with the greatest reduction in the coefficient of $a_2$ and produces a new list for the corresponding key number. Here again new transfers are possible and these may occur in different orderings. Elements from the programme can be found in the previous programmes. In order to find the optimal choice of the weights for the different lines in an inequality system the same routine as in the second programme is used. In order to go through all the possible transfers for the new key number a block of loops is used which looks similar to the one in the first programme. Also the procedure that determines the different orderings of the transfers by going through possible values of $\beta_1^{(4)}/\gamma_1$ and $\beta_1^{(4)}/\gamma_1$ is the same as in the second programme. When we couple an "old" inequality system together with a new one for the new key number we have to be careful and check that both inequality systems fit together. When we reach the stage where no transfer reduces the second coefficient (in front of $a_2$) in the corresponding list we are finished with this system, since we cannot get more information about it. Then the whole inequality system is stored on a file. Actually this happens only twice. The optimal bases are found in that way.
the transfer with the largest reduction in the coefficient in $a_2$. Then the corresponding list is determined. Possible transfers are tried out and all the possible orderings of these transfers are gone through. Suiting values for beta14 and beta13 are considered. They determine all possible orderings. In this way much fewer cases are considered than in the case where all possible combinatorial combinations are examined.

Here the variables used in the programme are defined.

```python
import math;
w=[None] * 14;g=[None] * 7;re=[None] * 17;rr=[None] * 17;RR=[None] * 17;A=[None] * 5200;
alin=[None] * 17; blin=[None] * 17;betalin=[None] * 17;t=[None] * 7;lin=[None]*7;
a=[None]*17; b1=[None]*17;bet1=[None]*17;kappa=[None]*50;
xend=[None] * 51; yend=[None]*51;ord=[None] * 11; s3f=[None] * 51; s4f=[None] * 51; s2f=[None] * 51;
s3=[None] * 51; s4=[None] * 51; s2=[None] * 51; f=[None] * 51; e=[None] * 51;
cv=[None] * 51;av=[None] * 51; bv=[None] * 51; betav=[None] * 51; Cv=[None] * 51; Av=[None] * 51; Bv=[None] * 51; Betav=[None] * 51; Ev=[None] * 51; Fv=[None] * 51;
numb=[None] * 51;length=[None]*7;S4=[None] * 51; S3=[None] * 51; S2=[None] * 51; ener=[None]*60; s4ff=[None]*51; s3ff=[None]*51; s2ff=[None]*51;
huskx=[None]*51; y=[None]*16; maks=[None]*7; SS='';SSS=''; ener1=ener2=L=x1=x2=x3=x4=x5=x6=x7=x8=x9=x10=x11=x12=0;
x13=x14=x15=plass=tell=i=antall=beta=Nev=k=j=repet=n=h=0;
p1=p2=sum=maxj=0;
TELLOPP=t1=b13=b14=ii=teller=E=F=zahl=zahltotal=bl=behag=0;
bhg1=bhg2=jj=0;
ihuks=ihuks=0;zzz=0; s2[0]=-1; s3[0]=0; s4[0]=0; opta=0.0; optb=0.0;
husk=huskb=huskc=a=b=c=opt=overtotal=total=0.0;
eps=0.0000001; counter1=0; opt=10; opt1=10; bound=2.008;
Er=[None]*800;
j=0;
while (j<800):
    Er[j]=[None]*31;
j+=1;
```
Here we choose the interval number \( p \) and the maximal total weight \( N \). Usually \( N = 12 \) is sufficient. The names of the input file and the output files are computed automatically. The input file is read and the inequality systems are displayed on the screen.

\[ p=4; \text{antblok}=4; m=0; M=0; N=12; \text{niv}=3; \text{controll}=0 \]

```python
filename="data"+str(p)+"niv"+str(niv+1)+".txt";
filename1="ferdig"+str(p)+"niv"+str(niv+1)+".txt";

ff=open("data"+str(p)+"niv"+str(niv)+".txt");

A = [];
for line in ff: # read rest of lines
    A.extend([[int(x) for x in line.split()]])
ff.close();

Here the \( s_3 \) values are computed according to the choice of \( s_4 \) for transfers from \( A \).

\[ g[0]=-1; \]
\[ i=1; \]
while( i<7 ):
    \[ g[i]=\text{math.floor}(i*w[p]+\text{eps});i+=1; \]

The different transfers that may occur in a list are ordered. First we have the transfers \( r_1, r_2, r_3, r_4, r_5, r_6 \in A \). Then we have \( q_1, q_2, q_3, q_4, q_5, q_6 \in B^* \) and in the end we have \( d_1, d_2, d_3 \in D^* \) and finally we get \( q \in E \). The values \( a_1, b_1 \) and \( \text{beta}1 \) represent the coefficients of \( \gamma_3, \gamma_2 \) and \( \beta_2^{(4)} \) in an optimal representation of the actual key number.

\[ a1=[\text{None}]*8; b1=[\text{None}]*8; \text{beta}1=[\text{None}]*8; \]
\[ j=0; \]
while ( j<7 ):
    \[ a1[j]=\text{None}]*18; b1[j]=\text{None}]*18; \text{beta}1[j]=\text{None}]*18; \]
    \[ j+=1; \]
\[ i=0; \]
while (i<7):
    j=1;
while (j<=16):
    if (j<7):
        a1[i][j]=1+j; b1[i][j]=-g[i]+g[j]; bet1[i][j]=i-j;
    if (j>6) and (j<13):
        a1[i][j]=1+j-6; b1[i][j]=-g[i]+g[j-6]+1; bet1[i][j]=i-j+6;
    if (j>12) and (j<16) :
        a1[i][j]=1; b1[i][j]=-g[i]+j-12; bet1[i][j]=i;
    j+=1;
    a1[i][16]=2; b1[i][16]=-g[i]+2; bet1[i][16]=i-1;
i+=1;

In the following lines the procedure that computes the coefficient from an inequality
\[ \epsilon_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta \]
according to (43) is presented. The procedure also contains an option for "pleasant" bases \(A_3\). In this case we may assume
\[ \gamma_3 + \gamma_2 + \gamma_1 \leq h + \delta, \]
which is a strong tool and the corresponding inequality systems often have small coefficients.

def kfbehag (a,b,c,behag):
    r = (3*(b+c)-2)/4; t = (b+c-4*b*c)/2; xx = math.sqrt(r*r+t)-r;
    if (xx >0):
        x=xx;
    else:
        x=0;
    opt=math.pow(1+ x,4)/(a*(b+x)*(c+x));
    if behag==1:
        h=1;
    while (h<=50):
        x1=h/50;
        mid=(1+x1)*(1+x1)*(1+x1)*((a+x1)*(b+x1)*(c+x1))/((1+x1)*(1+x1)*((a+x1)*(b+x1)*(c+x1)));
        if mid<opt:
            opt=mid;
        h+=1;
    return opt;
Since $\beta_2^{(4)}$ enters many of the inequalities but we need an expression containing $\gamma_2$ we here present a procedure that replaces $\beta_2^{(4)}$ with the upper or the lower bound according to the sign of the prefactor $\text{antbet}$.

```python
def Betval (antbet) :
    if (antbet>0) :
        return w[p];
    else :
        return w[p+1];
```

Here a procedure is presented that compares two lists. It picks two transfers one from each list and combines them using different weights like in (52). A coefficient is computed. If the coefficient is below 2.008 this can help to reduce the number of cases to consider.

```python
def kfs(rr,ind1,RR,ind2):
    M=10;j=1;opts=10;opt=10;
    while j<17 :
        i=1;
        while i<17:
            if rr[j]==1:
                i=1;
                while i<17:
                    if RR[i]==1:
                        jj=1;
                        while jj<=M:
                            aa=(jj*a1[ind1][j]+(M-jj)*a1[ind2][i])/M;
                            bb=(jj*b1[ind1][j]+(M-jj)*b1[ind2][i])/M;
                            antbet=(jj*bet1[ind1][j]+(M-jj)*bet1[ind2][i])/M;
                            bb=bb+antbet*Betval(antbet);
                            opt=kfbehag(aa,bb,0,0);
                            if opt<opts:
                                opts=opt;
                            jj+=1;
                        i+=1;
                    j+=1;
                return opts;
```

Here a procedure is presented that generates the different orderings of a set of transfers. In the start the orderings get their preliminary ordering. There is space for 10 transfers. Usually only some of them occur. The longest list observed contained 8 transfers (interval 6).
def transf(rr,s2,s3,s4,s2before,s3before,s4before,s2after,s3after,s4after) :
Erf=[None]*31;
i=1;tell=0;
while(i<=16):
    if (rr[i]==1) :
        tell=tell+1;
        if (i<7):
            s4f[tell]=i;s3f[tell]=g[i];
        if (i>6) and (i<13):
            s4f[tell]=i-6;s3f[tell]=g[i-6]+1;
        if (i>12) and (i<16) :
            s4f[tell]=0;s3f[tell]=i-12;
        if (i==16) :
            s4f[tell]=1;s3f[tell]=2;
        i+=1;
L=tell+1;
print("Lnge ",L);
    j=L;
    while (j<=10):
        s4f[j]=0;s3f[j]=0;
        j+=1;
    t1=0;
    if L==1:
        t1=t1+1;
    j=1;
    while (j<=30) :
        Er[t1][j]=0;
        j+=1;

Here we go through possible values of $\beta_1^{(4)}/\gamma_1$ and $\beta_1^{(3)}/\gamma_1$ in order to determine the ordering of the transfers and the $s_2$-values of the transfers.

if L>1:
    p1=29;p2=31;
b14=1;
    while (b14<p1):
        b13=1;
        while (b13<p2):
            Here we compute the actual $s_2$-values of the transfers.
j=1;
while (j<=10):
s2f[j]=math.floor(s3f[j]*b13/p2+s4f[j]*b14/p1);
j+=1;

We now look at the transfer that has got the number \( j \) in the preliminary ordering. We compare the reduction of the constant term with all other reductions of the constant terms from the list. Every time we find a transfer with a greater reduction of the constant term our transfer is moved one position further. When we have gone through all transfers we have found the correct position of the transfer \( j \) under consideration. Thus we find the position of all transfers of the list. Now the reduction of the constant terms are ordered in decreasing order thus the constant terms of all lines are non-negative. Since every reduction is below \( \gamma_1 \) by definition, we know that the constant terms are between 0 and \( \gamma_1 \). But we have to check whether this is the case where two lists meet. We have to check whether the first constant term on the new list where the transfer with the maximal \( a_2 \) reduction stood, is between 0 and \( \gamma_1 \). The same check has to be performed at the end of the new transfers.

j=1;
while (j<L) :
plass=1;
i=1;
while (i<L):
    if p1*(s3f[j]-s3f[i])*b13+p2*(s4f[j]-s4f[i])*b14-(s2f[j]-s2f[i])*p1*p2<0 :
        #kappaf[i]-s2f[i]>kappaf[j]-s2f[j] :
            plass=plass+1;
i+=1;
    ord[plass]=j;
j+=1;
j=L;
while (j<=10):
    ord[j]=j;
j+=1;

Here the transfers are listed according to the actual ordering.

j=1;
while (j<=10) :
h=3*(j-1);k=ord[j];Erf[h+1]=s4f[k];Erf[h+2]=s3f[k];Erf[h+3]=s2f[k];
s4ff[j]=s4f[k];s3ff[j]=s3f[k];s2ff[j]=s2f[k];
j+=1;
Here we determine whether an ordering of the transfers has occurred before or not.

```
repet=0;
ii=1;

while (ii<=t1):
    sum=0;j=1;
    while (j<=30):
        if (Er[ii][j]==Erf[j]):
            sum=sum+1;
        j+=1;
    if (sum==30) :
        repet=1;
    ii+=1;
```

Here we check if the constant terms of the first and the last new lines in the list are between 0 and $\gamma_1$.

```
check1=check2=1;check3=1;
if L>1:
    ener1=p1*(s3before-s3ff[1])*b13+
p2*(s4before-s4ff[1])*b14-(s2before-s2ff[1])*p1*p2;
    ener2=p1*(s3ff[L-1]-s3after)*b13+
p2*(s4ff[L-1]-s4after)*b14-(s2ff[L-1]-s2after)*p1*p2;
    if (ener1<0) or (ener1>p1*p2) :
        check1=0;
    if (ener2<0) or (ener2>p1*p2) :
        check2=0;
```

Here we check if the constant terms of the previous list are between 0 and $\gamma_1$.

Since we here choose new transfers by varying $\beta_1^{(4)}/\gamma_1$ and $\beta_2^{(3)}/\gamma_1$ it can happen that the constant terms in the old list may be inconsistent.

```
j=1;
while j<=(niv+1)*10:
    ener[j]=cv[j]*p1*p2+e[j]*b14*p2+f[j]*b13*p1;
    if (ener[j]>p1*p2) or (ener[j]<0):
        check3=0;
    j+=1;
```

If the ordering of the transfers is new (no repetition) and the additional checks are passed, the ordered set of transfers is stored and displayed on the screen.
if ((repet==0) and (check1==1) and (check2==1) and (check3==1)) or (t1==0) :
    t1=t1+1;
    j=1;
    while (j<=30) :
        Er[t1][j]=Erf[j];
        j+=1;
    if kontrollEr==1 :
        b13+=1;
        b14+=1;
    return(t1);

The following procedure administrates the optimization process by choosing weights $x_1, x_2, \ldots, x_{15}$. The total weight is always $\leq 12$. This value showed up to be sufficient during the first experiments. In order to reduce the number of lines in the inequality system to be considered the programme determines if any of the lines are repetitions or give less information as others. If all entries in one line but the one belonging to $a_2$ are equal only the one with the greater prefactor for $a_2$ has to be considered. The weight for the other line is put to 0. This reduces the amount of work considerably. The number $E$ gives us the accumulated amount of the $\beta^{(4)}_1$ whilst $F$ stands for the accumulated amount of $\beta^{(3)}_1$. If $F = E = 0$ the weight combination makes $\beta^{(4)}_1$ and $\beta^{(3)}_1$ disappear and we have an inequality of the kind

$$\epsilon_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta$$

when we consider the bounds for $\beta^{(4)}_2$. For this situation (43) gives us a way to compute the coefficient. When $c \leq 0$ and either $E = 0$ and $F = -1$ or $E = -1$ and $F = 0$ the ”constant term” in the weighted average inequality is negative, which is impossible. In this case we put the coefficient 2. When $c > 0$ and either $E = 0$ and $F = -1$ or $E = -1$ and $F = 0$ the ”constant term” in the weighted average inequality is $\geq (c - 1)\gamma_1$ and we may use $c - 1$ as the number of $\gamma_1$ in the formula when computing the coefficient. If all coefficients for all distributions of weights are above 2.008 then the inequality system is stored on a file and the programme proceeds to the next case on the input file.

def optim(av,bv,cv,betav,e,f,N):
    j=1;
    while (j<=(niv+2)*10):
        xend[j]=1;
if ((av[j]==1) and (bv[j]==0) and (betav[j]==0) and
(s4[j]==0) and (s3[j]==0) and (s2[j]==0) and
(e[j]==0) and (f[j]==0) and(cv[j]==0)) :
    xend[j]=0;
i=1;
while i<=10*(niv+2) :
    if (xend[j]==1) and not(i==j) and (e[j]==e[i]) and
(f[j]==f[i]) and (cv[j]==cv[i]) and
(bv[j]*120+betav[j]*(w[p]+w[p+1])*60<bv[i]*120+
betav[i]*(w[p]+w[p+1])*60) and (av[j]<=av[i]) :
        xend[j]=0;
i+=1;
j+=1;
j=1; zzz=0;
while (j<=(niv+2)*10) :
    if xend[j]==1:
        zzz=zzz+1; yend[zzz]=1; Av[zzz]=av[j]; Bv[zzz]=bv[j]; Cv[zzz]=cv[j];
        Betav[zzz]=betav[j]; Ev[zzz]=e[j]; Fv[zzz]=f[j];
        S4[zzz]=s4[j]; S3[zzz]=s3[j]; S2[zzz]=s2[j]; numb[zzz]=j;
        if kontroll==1 :
            print("Optimcheck",S4[zzz],S3[zzz],S2[zzz],
            Av[zzz],Bv[zzz],Cv[zzz],Betav[zzz],Ev[zzz],Fv[zzz],
            numb[zzz]);
        j+=1;
j=zzz+1;
while (j<=(niv+2)*10) :
    yend[j]=0; Av[j]=1; Bv[j]=Cv[j]=Betav[j]=0;
j+=1;
behag=0; bhg1=0; bhg2=0;
j=1;
while (j<=(niv+2)*10) :
    if ((e[j]==1) and (f[j]==1) and (cv[j]==0)) or
    ((e[j]==0) and (f[j]==0) and (cv[j]==1)):
        behag=1; bhg1=1;
i=1;
while (i<=(niv+2)*10):
    if (e[j]==e[i]) and (f[j]>f[i]) and (cv[j]==cv[i])
    and (not((e[i]==0) and (f[i]==0))) and (not((e[j]==0) and (f[j]==0)))) :
        behag=1; bhg2=1; ihusk=i; jhusk=j;
i+=1;
j+=1;
x1=0; total=10;
while (x1<=N) and (total>bound):
x2=0;
while (x2<=yend[2]*(N-x1)) and (total>bound):
x3=0;
while (x3<=yend[3]*(N-x1-x2)) and (total>bound):
x4=0;
...
x14=0;
while (x14<=yend[14]*(N-x1-x2-x3-x4-x5-x6-x7-x8-x9-x10
-x11-x12-x13)) and (total>bound):
x15=0;
while (x15<=yend[15]*(N-x1-x2-x3-x4-x5-x6-x7-x8-x9-x10
-x11-x12-x13-x14)) and (total>bound):
Nev=0; E=0; F=0; a=0; antall=0; b=0; c=0;
i=1;
while (i<=15):
    Nev=Nev+y[i]; E=E+Ev[i]*y[i]; F=F+Fv[i]*y[i];
    a=a+Av[i]*y[i]; antall=antall+Betav[i]*y[i];
    b=b+Bv[i]*y[i]; c=c+Cv[i]*y[i];
    i+=1;
opt=10;
T1=((Nev>0) and (F>=0) and (F==0));
T2=((c>=0) and (F==0) and (E==0));
T3=((c>=1) and (((F==0) and (E==-1)) or ((F==-1) and (E==0))));
T4=((c<=0) and (((F==0) and (E==-1)) or ((F==-1) and (E==0))));
if T1 and (T2 or T3 or T4) and (total>2):
    if T3 and not(T2) and not(T4):
        opt=kfbehag(a/Nev,(b+antall*Betval(antall))/Nev,
                   (c-1)/Nev,behag);
    if T2 and not(T3) and not(T4):
        opt=kfbehag(a/Nev,(b+antall*Betval(antall))/Nev,c/Nev,behag);
    if T4:
        opt=2;
if (opt<total):
    total=opt;
i=1;
while (i<=(niv+2)*10):
    huskx[i]=0;
    i+=1;
i=1;
while i<=15:
    huskx[numb[i]]=y[i];
    i+=1;
    x15+=1;
    x14+=1;
    ...
    x4+=1;
    x3+=1;
    x2+=1;
    x1+=1;
    return(total);

Here the programme starts.

The input file is read and the information is stored in the matrix A. Then the matrix is displayed on the screen.

i=0;
while (i<len(A)) :
    j=0;
    while (j<len(A[i])) :
        print (A[i][j], end=" ", flush=True);
        j+=1;
        print (" ");
    i+=1;
    print("Before the loops");

Here the output files are prepared.

SS+=str(0)+" ";SS+=str(0)+" ";SS+=str(0)+"\n";
SSS+=str(0)+" ";SSS+=str(0)+" ";SSS+=str(0)+"\n";

behag=0;
bl=1;

The variable bl is the number of the system under consideration. Here bl runs from 1 to antblok, the total number of systems. The input file information is displayed on the screen one system at a time.

while (bl<=antblok) :
    print("-------------------------");zs=(bl-1)*(11*(niv+1));
j=zs+1;
while (j<=zs+niv+1) :
    print(A[j][1],A[j][2],A[j][3],A[j][4],A[j][5],A[j][6],A[j][7],A[j][8],
A[j][9],A[j][10],A[j][11],A[j][12],A[j][13],A[j][14],
A[j][15],A[j][16],A[j][17]);
    j+=1;

i=1;
while (i<=10*(niv+1)):
    z=zs+niv+i;
    if kontroll==1 :
        print(A[z][1],A[z][2],A[z][3],A[z][4],A[z][5],
        A[z][6],A[z][7],A[z][8],A[z][9],A[z][10]);
    i+=1;
    print("+++++++");

Here the input information is decoded as transfers $(s_4, s_3, s_2)$ - in reverse order - and coefficients of the representations of the keynumbers

$$\epsilon_4 + av\gamma_3 + bv\gamma_2 + betav\beta_2^{(4)} + cv\gamma_1 + e\beta_1^{(4)} + f\beta_1^{(3)} \leq h + \delta$$

are stored on separate vectors.

j=1;
while j<=(niv+1)*10 :
    GG=zs+niv+i;
    j+=1;

Here the transfer that gives the largest reduction of the second term is determined for every level. We also determine the number of the line in the list where this occurs.

maks[-1]=0;
jj=0;lengniv=1;
while jj<niv :
    j=1;MM=0;maks[jj]=0;Red=0.0;length[jj]=1;
    while (j<=10):
        s4n=A[zs+niv+1+jj*10+j][1];s3n=A[zs+niv+1+jj*10+j][2];
        if (s4n>0) or (s3n>0) :
length[jj]=length[jj]+1;
if (s4n>0) and (s3n==g[s4n]):
    Red=s4n*(w[p+1]+w[p])/2-s3n;
if (Red>MM):
    MM=Red;maks[jj]=s4n;lin[jj]=j;
j+=1;
jj+=1;
maxj=maks[niv];
print("Maxj und lin ",maxj,lin[niv]);

If maxj = 0 we have no transfer in the list that reduces the coefficient of $a_2$. Then we cannot get more information about this case and we store the inequality system on a file for finished cases.

if maxj==0 :
nn=1;print("System finished");
while nn<=niv+1:
    tt=0;print(A[zs+nn]);
    while (tt<=17) :
        SSS+=str(A[zs+nn][tt])+' '
        tt+=1;
    SSS+="\n";
nn+=1;

j=1;
while (j<=(niv+1)*10):
    print(av[j],bv[j],cv[j],betav[j],e[j],f[j],huskx[j]);
    SSS+=str(0)+' ';SSS+=str(s4[j])+' ';SSS+=str(s3[j])+' '
    SSS+=str(s2[j])+' ';SSS+=str(av[j])+' ';SSS+=str(bv[j])+'
    SSS+=str(cv[j])+' ';SSS+=str(betav[j])+' ';SSS+=str(e[j])+'
    SSS+=str(f[j])+' ';SSS+=str(huskx[j])+'\n';
    j+=1;

If maxj > 0 there is a transfer in the list that reduces the coefficient of $a_2$. Then we get more information about this case by determining the new list and expand the inequality system. First we determine the transfer that was used before the one with the maximal reduction of the $a_2$ term and the one that was used after. Then we determine the loop ends of the variables $r_i, q_i, d_i$ and $q$. If a transfer has been used in a list it cannot be among the new ones in a later list. In interval 12 we know $r_6 = r_5 = r_4 = 0$ in the main list and $r_6 = 0$ in the two following lists. We also have $r_5 = 0$ for the first list after the main list if $r_1 = 1$ in the main list.
if maxj>0:
s2before=-1; s3before=s4before=0;
HH=zs+niv+1+niv*10+lin[niv];
if (lin[niv]>1):
s3before=A[HH-1][2]; s4before=A[HH-1][1]; s2before=A[HH-1][3];
s4after=A[HH+1][1]; s3after=A[HH+1][2]; s2after=A[HH+1][3];

if controll==1:
    print("Maximal reduction",maxj,"Line number",lin[niv],"s3before","s4before","s3after","s4after");

j=1;
while ( j<17 ):
i=0; ub=0;
while (i<=niv) :
    if (A[zs+1+i][j]==1):
        ub=1;#used before
    i+=1;
Here we check whether a given transfer can be used for the actual key number. If not we put re[j] = 0 and the corresponding transfer will not be used in the list. We also check whether a given transfer together with one of the other transfers that are known to be used in the previous list may give a combined inequality that causes the coefficient to drop below 2.008. Then we need not to consider this transfer.

re[j]=1; opta=kfbehag(a1[maxj][j], b1[maxj][j]+bet1[maxj][j]*Betval(bet1[maxj][j]),0,0);
i=1;
while i<17 :
    RR[i]=0;
    i+=1;
RR[j]=1;
jj=0; optbb=10;
while jj<=niv :
    optb=kfs(RR,maxj,A[zs+jj+1], maks[jj-1]);
    if optb<optbb :
        optbb=optb;
    jj+=1;
if (opta<bound) or (ub==1) or (optbb<bound) :
    re[j]=0;
    j+=1;
j=1;
while j<17 :
    if (maxj+bet1[0][j])*(w[p]+w[p+1])*60-120*(g[maxj]-b1[0][j]+1)<120*eps :
        re[j]=0;
    j+=1;

Here we use the information from chapter 11 where we exclude certain transfers from consideration.

    if ((p==12) and (niv<=2)) or (niv==0) :
        re[6]=0;

    if (niv==0) and (A[zs+1][1]==1):
        re[5]=0;

Here we compute some of the magnitudes used in (58). Later on the missing magnitudes are determined and the formula is applied.

    if niv==0 :
        ss4=0;ss3=A[zs+1][13]+2*A[zs+1][14];j=1;
        while (j<=6):
            ss4+=j*(A[zs+1][j]+A[zs+1][j+6]);
            ss3+=(A[zs+1][j]+A[zs+1][j+6])*g[j]+A[zs+1][j+6];
            j+=1;
        j=1;L0=1;
        while j<17:
            L0=L0+A[zs+1][j];
            j+=1;
        if controll==1 :
            print("L0",L0,"ss4",ss4,"ss3",ss3);
        if controll==1 :
            print(" End of the loops ",re);

    behag=0;
r6=0;
while (r6<=re[6]):
r5=0;
while (r5<=re[5]):
r4=0;
while (r4<=re[4]):
r3=0;
while (r3<=re[3]):
r2=0;
while (r2<=re[2]):
    r1=0;
while (r1<=re[1]):
    q6=0;
while (q6<=re[12]):
    q5=0;
while (q5<=re[11]):
    q4=0;
while (q4<=re[10]):
    q3=0;
while (q3<=re[9]):
    q2=0;
while (q2<=re[8]):
    q1=0;
while (q1<=re[7]):
    d3=0;
while (d3<=re[15]):
    d2=0;
while (d2<=re[14]):
    d1=0;
while (d1<=re[13]):
    q=0;
while (q<=re[16]):
    L=1; j=1;
    while (j<17):
        L+=rr[j]; j+=1;
    optund=10;
    if (L>1):
        kk=0;
        while (kk<=niv):
            optundv=kfs(rr,maxj,A[zs+kk+1],maks[kk-1]);
            if (optundv<optund):
                optund=optundv;
                kk+=1;

Here we check whether the combination of two transfers from the new list can give us an average inequality with coefficient below 2.008.
opt1=10;
Here we use the method described in (58) in order to compute a better bound for the coefficient when two lists are given.

if niv==0 :
    ss4s=q;ss3s=d1+2*d2+3*d3+2*q;j=1;
    while (j<=6):
        ss4s+=j*(rr[j]+rr[j+6]);
        ss3s+=(rr[j]+rr[j+6])*g[j]+rr[j+6];
        j+=1;
    xx=0;opt2=10;
    while xx<=20 :
        x=xx/20;T=L0+(L-2)*x;talbet=ss4+x*(ss4s-(L+1)*maxj);
        a=1+(ss4+x*(ss4s-maxj))/T;
        b=(L0+ss3+(ss3s-2-(L+1)*g[maxj])*x-talbet*Betval(-talbet))/T;
        c=1/T;
        if opt2<opt1:
            opt1=opt2;
            xx+=1;
We proceed only when both of the coefficient bounds are above 2.008.

if (opt1>bound) and (optund>bound) :
    T1=transf(rr,s2,s3,s4,s2before,s3before,s4before,
s2after,s3after,s4after);
n=1;
    while n<=T1 :
        j=1;
        while j<=10 :
            TT=(niv+1)*10+j;PP=TT-10;
            if j<lin[niv]:
                s4[TT]=s4[PP];s3[TT]=s3[PP];s2[TT]=s2[PP];
                #Start of new list

Here the new list is constructed. First we use the same transfers as in the previous level until we come to the line where the transfer with maximal reduction of the second term stood.

    s4[TT]=s4[PP];s3[TT]=s3[PP];s2[TT]=s2[PP];
#Start of new list

Here we continue the new list with the new transfers ordered in the order that the procedure transf has computed.

if (j>=lin[niv]) and (j<lin[niv]+L-1) :
    jjj=(j-lin[niv])*3
    s4[TT]=Er[n][jjj+1];s3[TT]=Er[n][jjj+2];
    s2[TT]=Er[n][jjj+3];
    # New transfers
Here we continue the new list with the transfers from the previous list from the line after the crucial line with the maximal reduction of the second term.

\[
\text{if } (j>=\text{lin}[niv]+L-1) \text{ and } (j<=\text{length}[niv]-2+L) :
\]
\[
s4[TT]=s4[PP+2-L];s3[TT]=s3[PP+2-L];s2[TT]=s2[PP+2-L];
\]
# Rest of new list

Here we fill inn zeroes and ones until line 10 for the new list (dummy values).

\[
\text{if } (j>\text{length}[niv]-2+L) :
\]
\[
s4[TT]=0;s3[TT]=0;s2[TT]=0;\#\text{End of list}
\]

Here we determine the coefficients of the representations of the key numbers.

\[
\begin{align*}
\text{av}[TT] &= 1+s4[TT];\text{bv}[TT] = -g[\maxj]+s3[TT]; \\
\text{cv}[TT] &=-s2[TT-1]+s2[TT];\text{betav}[TT] = \maxj-s4[TT]; \\
\text{e}[TT] &= s4[TT-1]-s4[TT];\text{f}[TT] = s3[TT-1]-s3[TT]; \\
\text{huskx}[TT] &= 0;
\end{align*}
\]
if j==1:
\[
\text{cv}[TT]=1+s2[TT];\text{e}[TT]=-s4[TT];f[TT]=-s3[TT];
\]
if (j>\text{length}[niv]-2+L) :
\[
\text{av}[TT]=1;
\text{bv}[TT]=\text{cv}[TT]=\text{betav}[TT]=\text{e}[TT]=\text{f}[TT]=\text{huskx}[TT]=0;
\]

Here we run the optimization procedure in order to find a good coefficient.

\[
\text{optw=optim(}\text{av,bv,}\text{cv,}\text{betav,}\text{e,f,N);}\;
\text{print("Coefficient ",optw,"blocknumber ",bl);}\;
\]

if optw>bound :
\[
\text{counter1}+=1;
\]

Here we store the surviving inequality system on a file and display it on the screen.

\[
\text{print(counter1," ",rr,}\text{math.floor(10000*optw));}
\text{j=1;}
\text{while j<=(niv+2)*10 :}
\text{print(s4[j],s3[j],s2[j],av[j],bv[j],cv[j],}
\text{betav[j],e[j],f[j],huskx[j]);}
\text{j+=1;}
\text{nn=1;}
\text{while nn<=niv+1:}
\text{tt=0;print(A[zs+nn]);}
\]
while (tt<=17) :
    SS+=str(A[zs+nn][tt]) + " ";
    tt+=1;
    SS="\n";
    nn+=1;

kk=0;rr[0]=0;
while (kk<=16) :
    SS+=str(rr[kk]) + " ";
    kk+=1;
    SS+=str(math.floor(10000*optw)) + "\n";
    j=1;
    while (j<=(niv+2)*10):
        print(av[j],bv[j],cv[j],betav[j],e[j],f[j],huskx[j]);
        SS+=str(0) + " "; SS+=str(s4[j]) + " ";
        SS+=str(s3[j]) + " "; SS+=str(s2[j]) + " ";
        SS+=str(av[j]) + " "; SS+=str(bv[j]) + " ";
        SS+=str(cv[j]) + " "; SS+=str(betav[j]) + " ";
        SS+=str(e[j]) + " "; SS+=str(f[j]) + " ";
        SS+=str(huskx[j]) + "\n";
        j+=1;

    if (optw>total):
        total=optw;
        n+=1;
        q+=1;
        d1+=1;
        d2+=1;
        d3+=1;
        q1+=1;
        q2+=1;
        q3+=1;
        q4+=1;
        q5+=1;
        q6+=1;
        r1+=1;
        r2+=1;
        r3+=1;
        r4+=1;
        r5+=1;
        r6+=1;

    bl+=1;
print("Blocknumber ",bl," Maximal coefficient ",total);
print("Number of systems ",counter1);

ff=open(filename,"w");ff.write(SS);ff.close();

ff=open(filename1,"w");ff.write(SSS);ff.close();
14 Performing the plan

Having arrived at this point of the story we have to tell how to use the computer programme and how to use the intermediate results in order to complete the proof. First we choose the intervall number $p$ and run the first programme. In the table below we see an overview over the results we get.

<table>
<thead>
<tr>
<th>Interval $I_p$</th>
<th>Largest coefficient bound</th>
<th>Total number of cases</th>
<th>Number of cases with coefficient bound $&gt; 2.008$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1 = (0, \frac{1}{6})$</td>
<td>2.0</td>
<td>1024</td>
<td>0</td>
</tr>
<tr>
<td>$I_2 = (\frac{1}{6}, \frac{1}{4})$</td>
<td>2.152</td>
<td>512</td>
<td>8</td>
</tr>
<tr>
<td>$I_3 = (\frac{1}{4}, \frac{1}{3})$</td>
<td>2.368</td>
<td>256</td>
<td>15</td>
</tr>
<tr>
<td>$I_4 = (\frac{1}{3}, \frac{1}{2})$</td>
<td>2.596</td>
<td>128</td>
<td>14</td>
</tr>
<tr>
<td>$I_5 = (\frac{1}{2}, \frac{2}{3})$</td>
<td>2.410</td>
<td>256</td>
<td>9</td>
</tr>
<tr>
<td>$I_6 = (\frac{2}{3}, \frac{3}{4})$</td>
<td>2.776</td>
<td>512</td>
<td>65</td>
</tr>
<tr>
<td>$I_7 = (\frac{3}{4}, \frac{4}{5})$</td>
<td>2.297</td>
<td>256</td>
<td>20</td>
</tr>
<tr>
<td>$I_8 = (\frac{4}{5}, \frac{5}{6})$</td>
<td>2.596</td>
<td>256</td>
<td>24</td>
</tr>
<tr>
<td>$I_9 = (\frac{5}{6}, \frac{2}{3})$</td>
<td>2.555</td>
<td>512</td>
<td>30</td>
</tr>
<tr>
<td>$I_{10} = (\frac{2}{3}, \frac{1}{2})$</td>
<td>2.646</td>
<td>256</td>
<td>29</td>
</tr>
<tr>
<td>$I_{11} = (\frac{1}{2}, \frac{1}{3})$</td>
<td>2.776</td>
<td>128</td>
<td>22</td>
</tr>
<tr>
<td>$I_{12} = (\frac{1}{3}, 1)$</td>
<td>3.600</td>
<td>256</td>
<td>54</td>
</tr>
</tbody>
</table>

The sets of transfers belonging to each of these cases are stored on the files $res1, res2, ..., res12$. The result for interval $I_2$ is given in table 43. Here we show the result for interval $I_4$, since the optimal bases are found here. They belong to case 12.
Table 44. The cases in interval $I_4$

<table>
<thead>
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<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_5$</th>
<th>$r_6$</th>
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<th>$q_2$</th>
<th>$d_1$</th>
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</table>

Now in 11 of the 12 intervals we find sets of transfers like the ones from interval $I_4$ displayed above, where the average inequality does not produce a coefficient below 2.008. These sets are therefore potential candidates for the extremal $h$-range bases.

Now we choose an interval number $2 \leq p \leq 12$ and the second programme reads the file with the sets of transfers that could not be ruled out. Of course the transfers from such a set can occur in different orderings. The second programme therefore runs through all possible orderings. Here we use some number theoretic information, since the combinatorial number of orderings would be vast. The programme chooses values for $\beta_1^{(3)}/\gamma_1$ and $\beta_1^{(4)}/\gamma_1$ and determines the reduction of the constant term belonging to the actual transfers and puts them in order. Then the transfers have to occur in exactly this order in the list since the list was constructed using decreasing reductions of the constant term. We choose the following values: $\beta_1^{(3)}/\gamma_1 = k \cdot 29 \cdot \gamma_1$ for $k = 1, 2, \ldots, 28$ and $\beta_1^{(4)}/\gamma_1 = k \cdot 31 \cdot \gamma_1$ for $k = 1, 2, \ldots, 30$. By this choice we make sure that all the possible values of

$$\kappa = s_4 \beta_1^{(4)} + s_3 \beta_1^{(3)} - s_2 \gamma_1$$

are different and the transfers can be ordered. Once a combination of values for $\beta_1^{(3)}/\gamma_1$ and $\beta_1^{(4)}/\gamma_1$ is chosen, the ordering of the transfers follows. Given the ordering we now can determine the corresponding inequality system, describing the coefficient sums for the keynumbers. Here the programme chooses suitable weights for the lines in the list (total sum $\leq 12$) and determines the weighted average inequalities. Using these average inequalities a coefficient for the
$h$-range is computed. If this coefficient is below 2.008 the inequality system is discarded and the programme proceeds to the next system. If the coefficient is above 2.008 the system is stored on a file. In interval $I_2$ f.ex. only two systems survive. In interval $I_3$ 354 systems survive. In the third column in the table below you will find how many systems survive the weighted average method. Now the third programme reads one of these surviving systems and determines which transfer has the largest reduction of the second term (in front of $a_2$ compared to the regular representation of the key number). This transfer gives rise to the next key number and determines the next list. In order to produce this list the programme goes through the possible transfers and once it has chosen a set of transfers it goes through the possible orderings of these. In the end we get new or extended inequality systems. The weighted average procedure for these systems then finds the coefficient of the $h$-range and discards the inequality system if the coefficient is below 2.008. In the opposite case the system is filed and can be examined later on in the same way on a new level. Some systems have to go through 4 levels before they can be excluded. On every new level a new key number will be considered. If there are no transfers left in the list that reduce the second term we cannot get more information from this case and we have to finish our work filing the system on a special file. This only happens twice when the optimal bases are found. In interval $I_2$ the two systems do not survive to the next level i.e. that for all transfers that could represent the new key number the corresponding inequality system gives a coefficient below 2.008. No inequality system survives level 4 besides the optimal cases in interval $I_4$. One pair of bases made it up to the same level but there was an additional key number. When taking this additional key number into account the coefficient for the $h$-range dropped below 2.008.

Table 44. Levels

<table>
<thead>
<tr>
<th>$p$ Interval</th>
<th>$N_C$</th>
<th>$N_S(0)$ Level 0</th>
<th>$N_S(1)$ Level 1</th>
<th>$N_S(2)$ Level 2</th>
<th>$N_S(3)$ Level 3</th>
<th>$N_S(4)$ Level 4</th>
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<td>282</td>
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</tbody>
</table>
$p$ is the number of the interval.
$N_C$ is the number of sets of transfers with coefficient above 2.008.
$N_S(i)$ is the number of inequality systems on level $i$.

Since we only found two inequality systems with coefficient above 2.008 and these two systems correspond to the systems belonging to the Mossige and the Selmer basis we now have shown that there are no other bases with a greater coefficient neither are there other bases with the same coefficient. Therefore we have found the asymptotic optimal bases.
15 The optimal bases

In chapter 14 we saw that there were only two inequality systems where the upper bounds for the coefficient of the $h$-range could not be shown to be below 2.008. These systems were found in interval $I_4$. Here we present these systems in detail:

The first basis meets the following constraints:

Mainlist: $(0, 0, 2), (0, 1, 0), (0, 0, 1), (1, 0, 3), (0, 0, 0)$

\[
\begin{align*}
\epsilon_4 &+ 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} \leq h + \delta \\
\epsilon_4 &+ \gamma_3 + 2\gamma_2 + 2\beta_1^{(4)} - \beta_1^{(3)} \leq h + \delta \\
\epsilon_4 &+ 2\gamma_3 + \gamma_2 - \beta_2^{(4)} - \beta_1^{(4)} + \beta_1^{(3)} \leq h + \delta. \\
\epsilon_4 &+ 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} \leq h + \delta \\
\epsilon_4 &+ \gamma_3 + \gamma_2 + 3\beta_2^{(4)} - \gamma_1 \leq h + \delta
\end{align*}
\]

The list for $K(3):(0, 0, 2), (0, 1, 0), (0, 0, 1), (0, 0, 0)$

\[
\begin{align*}
\epsilon_4 &+ 3\gamma_3 + \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} \leq h + \delta \\
\epsilon_4 &+ \gamma_3 + \gamma_2 + 3\beta_2^{(4)} + 2\beta_1^{(4)} - \beta_1^{(3)} \leq h + \delta \\
\epsilon_4 &+ 2\gamma_3 + 2\beta_2^{(4)} - \beta_1^{(4)} + \beta_1^{(3)} \leq h + \delta. \\
\epsilon_4 &+ \gamma_3 + 3\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta
\end{align*}
\]

The list for $K(2):(0, 1, 0), (0, 0, 1), (0, 0, 0)$

\[
\begin{align*}
\epsilon_4 &+ \gamma_3 + \gamma_2 + 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} \leq h + \delta \\
\epsilon_4 &+ 2\gamma_3 + \beta_2^{(4)} - \beta_1^{(4)} + \beta_1^{(3)} \leq h + \delta. \\
\epsilon_4 &+ \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(4)} \leq h + \delta
\end{align*}
\]

The list for $K(1):(0, 1, 0), (1, 1, 2), (0, 0, 0)$

\[
\begin{align*}
\epsilon_4 &+ \gamma_3 + \gamma_2 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} \leq h + \delta \\
\epsilon_4 &+ 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} \leq h + \delta. \\
\epsilon_4 &+ \gamma_3 + \beta_2^{(4)} + 2\beta_1^{(4)} + \beta_1^{(3)} - \gamma_1 \leq h + \delta
\end{align*}
\]

Here the following weights were used. Main list: 0, 1, 2, 1, 0, $K(3)$-list: 0, 0, 0, 0, $K(2)$-list: 2, 0, 0, $K(1)$-list: 0, 0, 1, giving the average inequality

\[
\epsilon_4 + 12\gamma_3/7 + 7\gamma_2/7 + 2\gamma_1/7 \leq h + \delta
\]

with coefficient 2.00892857 according to (41) and (43). We see that this coefficient is slightly above the Mossige-Selmer-coefficient 2.0080397. This
incoherence can be explained. The choice of weights is not optimal. If we put $w = 873/1000$ and choose the weights: $1, 2, w, w + 1$ and $w$ for the "active" inequalities we get

$$
\epsilon_4 + 10962\gamma_3/6481 + 6654\gamma_2/6481 + 1854\gamma_1/6481 \leq h + \delta
$$

with coefficient $2.00803976$ according to (41) and (43). This result corresponds much better with the Mossige-Selmer-coefficient. Of course the computer programme could not find this solution since only total weights (sum of all weights) up to 12 were considered and all weights had to be integers. Nevertheless it is still amazing that the computer programme could exclude all other inequality systems even if it did not calculate the minimal upper bounds because of the mentioned weight restriction.

The second basis meets the following constraints:

Mainlist: $(1, 0, 3), (0, 0, 1), (0, 1, 0), (1, 0, 2), (0, 0, 0)$

$$
\epsilon_4 + 4\gamma_3 + \gamma_2 - 3\beta_2^{(4)} + 2\gamma_1 - 3\beta_1^{(4)} \leq h + \delta
$$
$$
\epsilon_4 + 2\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\beta_1^{(4)} - \gamma_1 \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + 2\gamma_2 + \beta_1^{(4)} - \beta_1^{(3)} \leq h + \delta.
$$
$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - 2\beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} + \beta_1^{(3)} \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + \gamma_2 + 2\beta_1^{(4)} - \gamma_1 \leq h + \delta
$$

The list for $K(3) : (0, 0, 1), (0, 1, 0), (1, 0, 2), (0, 0, 0)$

$$
\epsilon_4 + 2\gamma_3 + 2\beta_2^{(4)} + \gamma_1 - \beta_1^{(4)} \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + \gamma_2 + 3\beta_2^{(4)} + \beta_1^{(4)} - \beta_1^{(3)} \leq h + \delta
$$
$$
\epsilon_4 + 3\gamma_3 + \beta_2^{(4)} + \gamma_1 - 2\beta_1^{(4)} + \beta_1^{(3)} \leq h + \delta.
$$
$$
\epsilon_4 + \gamma_3 + 3\beta_2^{(4)} + 2\beta_1^{(4)} - \gamma_1 \leq h + \delta
$$

The list for $K(2) : (0, 0, 1), (0, 1, 0), (0, 0, 0)$

$$
\epsilon_4 + 2\gamma_3 + \beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + \gamma_2 + 2\beta_2^{(4)} + \beta_1^{(4)} - \beta_1^{(3)} \leq h + \delta.
$$
$$
\epsilon_4 + \gamma_3 + 2\beta_2^{(4)} + \beta_1^{(3)} \leq h + \delta
$$

The list for $K(1) : (1, 1, 2), (0, 1, 0), (0, 0, 0)$

$$
\epsilon_4 + 3\gamma_3 + \gamma_2 - \beta_2^{(4)} + 2\gamma_1 - 2\beta_1^{(4)} - \beta_1^{(3)} \leq h + \delta
$$
$$
\epsilon_4 + \gamma_3 + \gamma_2 + \beta_2^{(4)} + 2\beta_1^{(4)} - \gamma_1 \leq h + \delta.
$$
$$
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + \beta_1^{(3)} \leq h + \delta
$$
Here the following weights were used. Main list: 0, 1, 2, 1, 0, \( K(3) \)-list: 0, 0, 0, 0, \( K(2) \)-list: 0, 0, 2, \( K(1) \)-list: 1, 0, 0 again giving the average inequality

\[
\epsilon_4 + 12 \gamma_3 / 7 + 7 \gamma_2 / 7 + 2 \gamma_1 / 7 \leq h + \delta
\]

with coefficient 2.00892857 according to (41) and (43). Also this coefficient is slightly above the Mossige-Selmer-coefficient 2.0080397. This incoherence again can be explained. The choice of weights is not optimal. If we put \( w = \frac{827}{1000} \) and choose the weights: \( w, 2, 1, 1 + w \) and \( w \) for the "active" inequalities we get

\[
\epsilon_4 + 10962 \gamma_3 / 6481 + 6654 \gamma_2 / 6481 + 1854 \gamma_1 / 6481 \leq h + \delta
\]

again with coefficient 2.00803976 according to (41) and (43). This results corresponds much better with the Mossige-Selmer-coefficient. Of course the computer programme could not find this solution either because of the mentioned weight restriction.

In any of the two cases only five inequalities are sharp. We can see this because the corresponding weights are positive.

First base:

\[
\begin{align*}
\epsilon_4 + \gamma_3 + 2 \gamma_2 + 2 \beta_1^{(4)} - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + 2 \gamma_3 + \gamma_2 - \beta_2^{(4)} - \beta_1^{(4)} + \beta_1^{(3)} & \leq h + \delta. \\
\epsilon_4 + 4 \gamma_3 + \gamma_2 - 3 \beta_2^{(4)} + \gamma_1 - 2 \beta_1^{(4)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \gamma_2 + 2 \beta_2^{(4)} + \gamma_1 - \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + \beta_2^{(4)} + 2 \beta_1^{(4)} + \beta_1^{(3)} - \gamma_1 & \leq h + \delta
\end{align*}
\]

Second base:

\[
\begin{align*}
\epsilon_4 + 2 \gamma_3 + \gamma_2 - \beta_2^{(4)} + 2 \beta_1^{(4)} - \gamma_1 & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2 \gamma_2 + \beta_1^{(4)} - \beta_1^{(3)} & \leq h + \delta. \\
\epsilon_4 + 3 \gamma_3 + \gamma_2 - 2 \beta_2^{(4)} + \gamma_1 - 2 \beta_1^{(4)} + \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + \gamma_3 + 2 \beta_2^{(4)} + \beta_1^{(3)} & \leq h + \delta \\
\epsilon_4 + 3 \gamma_3 + \gamma_2 - \beta_2^{(4)} + 2 \gamma_1 - 2 \beta_1^{(4)} - \beta_1^{(3)} & \leq h + \delta
\end{align*}
\]

In addition we always have the constraint

\[
\gamma_2 + \gamma_1 \leq h + \delta.
\]

If we compare these inequality systems with the ones presented in [19] by Selmer we see that the first basis corresponds exactly to Selmer’s basis and the second one corresponds exactly to Mossige’s. Now Mossige and Selmer give lower bounds for the coefficients of the \( h \)-range of their bases whilst we here give upper bounds for these coefficients. Since we arrive at the same inequality
systems we now have found the extremal \( h \)-range (asymptotical) for four stamp denominations.

We have to comment on the methods used to find the inequality systems characterizing the optimal bases. Selmer and Mossige used the Kuhn Tucker conditions in order to find a maximum for their problem whilst we used Hofmeister’s idea on the arithmetic and geometric mean in addition to the method presented here in (41) and (43) to find the minimal upper bound. We may call this method the Hofmeister method in the sequel. In fact we shall show that these methods are equivalent which might be a bit surprising. Let us first recall some notions concerning the Kuhn Tucker conditions.

Given the real function \( f(x) = f(x_1, x_2, ..., x_n) \) on \( \mathbb{R}^n \). We want to maximize \( f \) under certain conditions \( g_i(x_1, x_2, ..., x_n) \leq 0 \) for \( i = 1, 2, ..., M \). Now the Kuhn Tucker conditions tell us that for a global maximum \( x^* \) there exist multipliers \( \lambda_i \geq 0 \) for \( i = 1, 2, ..., M \) such that

\[
\sum_{i=1}^{M} \lambda_i \nabla g_i(x^*) = \nabla f(x^*).
\]

This is a necessary condition for the global maximum \( x^* \). Sufficiency can be shown under additional conditions. The region of possible points has to be convex. This is the case in our problem since all constraints are linear and the object function \( f \) has to be "pseudo-concave". We shall not explain this notion here but we mention that our object function, being the product of some of the variables always is pseudo-concave as mentioned by Kolsdorf [8 bei Selmer] (H.-E. Kolsdorf, Ein Beitrag zur additiven Zahlentheorie, to appear.)

Now what do the Kuhn Tucker conditions mean in our problem, when we want to find extramal \( h \)-ranges? We put

\[
x_1 = \gamma_1/h, x_2 = \gamma_2/h, x_3 = \gamma_3/h, x_4 = \epsilon_4/h, x_5 = \beta_2^{(4)}/h, x_6 = \beta_1^{(4)}/h, x_7 = \beta_1^{(3)}/h.
\]

Now we are looking for \( x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \) fulfilling some linear conditions of the following form:

\[
g_{4,i}\epsilon_4 + g_{3,i}\gamma_3 + g_{2,i}\gamma_2 + g_{1,i}\gamma_1 + g_{5,i}\beta_2^{(4)} + g_{6,i}\beta_1^{(4)} + g_{7,i}\beta_1^{(3)} \leq h + \delta,
\]

for \( i = 1, 2, ..., M \). These constraints can now be written as

\[
g_{1,i}x_1 + g_{2,i}x_2 + g_{3,i}x_3 + g_{4,i}x_4 + g_{5,i}x_5 + g_{6,i}x_6 + g_{7,i}x_7 - 1 \leq 0.
\]

for \( i = 1, 2, ..., M \). These are the inequalities for the coefficient sums that describe the representability of the numbers in the interval \([\epsilon_4 - 1)a_4 + (\gamma_3 -2)a_3, (\epsilon_4 - 1)a_4 + (\gamma_3 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1]\). Usually we have several such inequalities. In addition we have

\[
x_1 + x_2 - 1 \leq 0
\]

which is of the same form. This inequality gets the index \( M + 1 \) and the multiplier \( w \). Of course the objective function is
\( f(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_4 \cdot x_3 \cdot x_2 \cdot x_1, \) describing the order of magnitude of the \( h \)-range. The Kuhn Tucker conditions now are seven partial differential equations, the last three of them are equal to zero on the right side, since the corresponding variables do not enter the objective function. Thus we have.

\[
\sum_{i=1}^{M} \lambda_i g_{ji,i} = 0
\]

for \( j = 5, 6, 7. \) Feeding this information back into the Kuhn Tucker conditions we can substitute

\[
\sum_{i=1}^{M} \lambda_i g_i(x^*)
\]

by a single linear constraint \( x_4 + ax_3 + bx_2 + cx_1 - 1 \leq 0 \) or

\[
e_4 + a\gamma_3 + b\gamma_2 + c\gamma_1 \leq h + \delta,
\]

where neither \( \beta_2^{(4)}, \beta_1^{(4)} \) nor \( \beta_1^{(3)} \) appear. This constraint we give the weight \( v. \) Then the Kuhn Tucker conditions read:

\[
v \nabla G(x^*) + w H(x^*) = \nabla f(x^*),
\]

where \( G(x) = x_4 + ax_3 + bx_2 + cx_1 - 1 \) and \( H(x) = x_1 + x_2 - 1. \) In detail these conditions read

\[
vc + w = x_4 \cdot x_3 \cdot x_2
\]

\[
v b + w = x_4 \cdot x_3 \cdot x_1
\]

\[
av = x_4 \cdot x_2 \cdot x_1
\]

\[
v = x_3 \cdot x_2 \cdot x_1
\]

From the last two equations we get

\[
x_3 = \frac{v}{x_1 x_2},
\]

\[
x_4 = \frac{av}{x_1 x_2} = ax_3.
\]

With help from the first two equations we get

\[
x_2 = \frac{x_4 x_2 x_1}{x_4 x_3 x_1} = \frac{av}{bv + w} x_3,
\]

\[
x_1 = \frac{x_4 x_3 x_1}{x_4 x_3 x_2} = \frac{bv + w}{cv + w} x_2 = \frac{av}{cv + w} x_3.
\]

Now from \( x_4 + ax_3 + bx_2 + cx_1 = 1 \) we get

\[
a x_3 + ax_3 + \frac{abv}{bv + w} x_3 + \frac{acv}{cv + w} x_3 = 1
\]

\[
a (2 + \frac{bv}{bv + w} + \frac{cv}{cv + w}) x_3 = 1
\]
\[ x_3 = \frac{(bv + w)(cv + w)}{a(2(bv + w)(cv + w) + bv(cv + w) + cv(bv + w))} \]  
\[ x_3 = \frac{(bv + w)(cv + w)}{a(4bcv^2 + 3(b + c)vw + 2w^2)}. \]

On the other hand we get from \( x_1 + x_2 = 1 \)

\[ \frac{av}{bv + w}x_3 + \frac{av}{cv + w}x_3 = 1 \]

\[ x_3 = \frac{(bv + w)(cv + w)}{av((b + c)v + 2w)}. \]

Now we have two different expressions for \( x_3 \) and may equate them:

\[ v((b + c)v + 2w) = 4bcv^2 + 3(b + c)vw + 2w^2 \]
\[ 0 = 2w^2 + (3(b + c) - 2)vw + (4bc - (b + c))v^2 \]
\[ 0 = 2 \left( \frac{w}{v} \right)^2 + (3(b + c) - 2) \left( \frac{w}{v} \right) + (4bc - (b + c)) \]

That means that the quotient of the weights fits into the same equation (44) as the optimal weight \( x \) in chapter 5 in (44). We use \( x \) for the positive solution of the quadratic equation and calculate the coefficient \( C \) of the \( h \)-range in to different ways.

\[ C = \frac{(1 + x)^4}{a(b + x)(c + x)} \]

from (43) and

\[ C = 4^4 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4 \]
\[ = 4^4 \cdot \frac{(cv + w)}{((b + c)v + 2w)} \cdot \frac{(bv + w)}{(b + c)v + 2w} \cdot \frac{(bv + w)(cv + w)}{av((b + c)v + 2w)} \cdot \frac{(bv + w)(cv + w)}{v((b + c)v + 2w)} \]
\[ = 4^4 \cdot \frac{(cv + w)^3(bv + w)^3}{av^2((b + c)v + 2w)^4} = 4^4 \cdot \frac{(c + w/v)^3(b + w/v)^3}{a((b + c) + 2w/v)^4} = 4^4 \cdot \frac{(c + x)^3(b + x)^3}{a((b + c) + 2x)^4} \]

from the Kuhn Tucker objective function. Equating the two expressions for the coefficient gives

\[ \frac{(1 + x)^4}{a(b + x)(c + x)} = 4^4 \cdot \frac{(c + x)^3(b + x)^3}{a((b + c) + 2x)^4} \]

or

\[ (1 + x)^4((b + c) + 2x)^4 = 4^4(c + x)^4(b + x)^4, \]

which again is equivalent to

\[ (1 + x)((b + c) + 2x) = 4(c + x)(b + x) \]

and thus to (44). Thus we have shown that the Kuhn Tucker conditions give the same result as the Hofmeister method and the methods are equivalent.
16 Five stamp denominations

Here we gather some ideas about the extremal $h$-range in the case of five stamp denominations. We try to mimic the ideas in chapter XYZ and find bounds for the number of transfers used in the optimal case. We shall see that this amount grows rapidly and we cannot expect to cover all cases as we did for $k = 4$. From Rødseth’s general upper bound (5) we know

$$T_5 \leq \frac{(k-1)^{k-1}}{(k-1)!} = \frac{4^4}{4!} = \frac{32}{3} = 9.666...$$

In [8]) Kolsdorp presets a sequence of bases $A_5(h)$, with $n_h(A_5(h)) \geq 3.06(h/5)^5$. This means that $3.06 \leq T_5 \leq 9.666....$

We look at a parameter basis $A_5 = A_5(h)$. If $n_h(A_5(h)) < 3.06(h/5)^5 + O(h^4)$, the basis cannot be an extremal one. Therefore we consider only bases $A_4(h)$ with

$$n_h(A_5(h)) \geq 3.06(h/5)^5 + O(h^4). \quad (74)$$

We use the normal form (2), and write

$$n_h(A_5(h)) = \epsilon_5 a_5 + \epsilon_4 a_4 + \epsilon_3 a_3 + \epsilon_2 a_2 + \epsilon_1 \quad (75)$$

for the regular representation of the $h$-range of a given parameter basis $A_5(h)$.

Using the Hofmeister method from chapter 5 we can show that if

$$\epsilon_5 + a\gamma_4 + b\gamma_3 + c\gamma_2 + d\gamma_1 \leq h + \delta \quad (76)$$

for positive constants $a, b, c, d \in \mathbb{R}$ and $\delta \in \mathbb{R}$ then

$$n_h(A_5(h)) < (\epsilon_5 + 1)a_4 \leq \epsilon_5\gamma_4\gamma_3\gamma_2\gamma_1 \leq \frac{1}{abcd} \left(\frac{h + \delta}{5}\right)^5 + O(h^4) = \frac{1}{abcd} \left(\frac{h}{5}\right)^4 + O(h^4). \quad (77)$$

This means that if we establish an inequality (76) where $\frac{1}{abcd} < 3.06$, then the sequence of bases $A_5(h)$ cannot be extremal and can therefore be excluded from further consideration. Now given such an equality (76), we may refine the result we can get from (77) by additional information. In the chapter 12 about $s > 0$ we considered two key numbers $R = (\gamma_2 - 2)a_3 + (\gamma_2 - 2)a_2 + \gamma_1 - 1$ and $S = (\gamma_2 - 2)a_3 + (\gamma_2 - 2)a_2 + \beta_1^{(4)} - 1$. From the minimal representations of these two numbers we could draw some information. One of the following inequalities has to hold (68):

$$\gamma_3 + \gamma_2 + \gamma_1 \leq h + 5.$$

or (69)

$$\gamma_3 + 3\gamma_2 \leq h + \delta,$$
or (70)
\[ \gamma_3 + 3\gamma_2/2 + \gamma_1/2 \leq h + \delta. \]
We always have
\[ \gamma_2 + \gamma_1 \leq h + 3. \]
We now combine our inequality (76)
\[ \epsilon_5 + a\gamma_4 + b\gamma_3 + c\gamma_2 + d\gamma_1 \leq h + \delta \]
with each of the new inequalities by non-negative weights \( x \geq 0, y \geq 0 \) and \( z \geq 0 \) and get
\[
\begin{align*}
\epsilon_5 + a\gamma_4 + (b + x)\gamma_3 + (c + x)\gamma_2 + (d + x)\gamma_1 & \leq (1 + x)h + \delta \\
\epsilon_5 + a\gamma_4 + (b + y)\gamma_3 + (c + y + z)\gamma_2 + z\gamma_1 & \leq (1 + y + z)h + \delta \\
\epsilon_5 + a\gamma_4 + (b + y)\gamma_3 + (c + 3y/2 + z)\gamma_2 + (y/2 + z)\gamma_1 & \leq (1 + y + z)h + \delta
\end{align*}
\]
(78)
Now we run through the interval \([0, 2]\) for the weights and register the least coefficient for the \( h \)-range for each of the three alternatives.

\[ Q_1 = \frac{(1 + x)^5}{a(b + x)(c + x)(d + x)}, \]
\[ Q_2 = \frac{(1 + y + z)^5}{a(b + y)(c + 3y + z)z}, \]
\[ Q_3 = \frac{(1 + y + z)^5}{a(b + y)(c + 3y/2 + z)(y/2 + z)} \]
The largest of these three coefficients then gives us an upper bound for the coefficient, since the cases (68), (69) and (70) cover all situations. Now we want to find the optimal representations for the numbers in the interval \([(\epsilon_5 - 1)a_5 + (\gamma_4 - 2)a_4, (\epsilon_5 - 1)a_5 + (\gamma_4 - 2)a_4 + (\gamma_3 - 2)a_2 + (\gamma_2 - 2)a_2 + \gamma_1 - 1]\).
If we use a transfer \((s_2, s_3, s_4, s_5)\) with \( s_5 \geq 17 \) we get
\[ \epsilon_5 + (1 + s_5)\gamma_4 = \epsilon_5 + 18\gamma_4 \leq h + \delta \]
giving a coefficient 3.04 according to the method presented above. Here we have \( Q_1 = 1.60 \) for \( x = 1.5, Q_2 = 3.04 \) for \( y = 0.9, z = 0.6 \) and \( Q_3 = 2.20 \) for \( y = 1.4, z = 0.1 \). Thus we know \( s_5 \leq 16 \). If we use a transfer with \( s_5 = 16 \) we get \( \epsilon_5 + 17\gamma_4 \leq h + \delta \) and we may ask whether there could be more than one transfer with \( s_5 = 16 \). If there were several such transfers they would differ in \( s_4 \) or \( s_3 \) and we would get
\[ \epsilon_5 + 17\gamma_4 + \gamma_3 \leq h + \delta \]
or
\[ \epsilon_5 + 17\gamma_4 + \gamma_2 \leq h + \delta. \]

Both of them imply a coefficient below 3.06 according to the method above. Thus there can only be one transfer with \( s_5 = 16 \). In list 45 below we go through different values of \( a, b, c \) and \( d \) and find coefficients below 3.06. In this manner we can find out whether there are more than one transfer with given \( s_5 \) and \( s_4 \). For \( s_5 = 12 \) and \( s_4 = 0 \) and \( s_3 = 1 \) the coefficient is not below 3.06 so here we have two transfers with \( s_5 = 5 \) and \( s_4 = 0 \) differing in \( s_3 \). By the same means we can find the number of transfers belonging to any given pair \( (s_4, s_5) \). Thus we get an overview over the possible transfers.
<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>Coefficient inequality</th>
<th>Coeff.</th>
<th>Number of transfers with given $s_5$ and $s_4$</th>
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Table 45 B Continuation

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<td>$\epsilon_5 + \gamma_4 + 10\gamma_3 \leq h + \delta$</td>
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<td>3.04</td>
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</table>

Alltogether 158 transfers have to be considered. Of course this univers of transfers is too vast to be examined in the same way as for $k = 4$. If this case should be manageable even for a computer we have to reduce this number considerably before we can try to use the same methods as presented here.

References


