Separating and joining variables in monomial ideals

Gunnar Fløystad

March 20, 2022

Highlights

1. Separations of monomial ideals
2. Polarizations of artinian monomial ideals

Highlights

1. Separations of monomial ideals
2. Polarizations of artinian monomial ideals

3 Focus

Triangulations of polygons

Highlights

1. Separations of monomial ideals
2. Polarizations of artinian monomial ideals

3. Focus

Triangulations of polygons

4. Monomial ideals associated to trees
5. $\rightsquigarrow$ stacked simplicial complexes

Separating a monomial ideal

- $I = (x_1^2, xy, y^2) \subseteq k[x,y] = S$, Resolution $I \leftarrow S(-2)^3 \leftarrow S(-3)^2$
- $J = (x_1x_2, x_1y, y^2) \subseteq k[x_1,x_2,y] = T$, Resolution $J \leftarrow T(-2)^3 \leftarrow T(-3)^2$
- $\frac{k[x,y]}{I}$ is a quotient of $\frac{k[x_1,x_2,y]}{J}$ – by dividing out by $x_2 - x_1$.

  – $x_2 - x_1$ is a non-zero divisor (regular element) in $\frac{k[x_1,x_2,y]}{J}$
- $J_1 = J = (x_1x_2, x_1y, y^2)$ is a separation of $I = (x_1^2, xy, y^2)$

Polarization

- Separate further to $J_2 = (x_1x_2, x_1y_1, y_1y_2) \subseteq k[x_1, x_2, y_1, y_2] = T_2$

  Resolution $J_2 \leftarrow T_2(-2)^3 \leftarrow T_2(-3)^2$

- $\frac{k[x_1,x_2,y]}{J_1}$ is a quotient of $\frac{k[x_1,x_2,y_1,y_2]}{J_2}$
  - by dividing out by $y_2 - y_1$
  - $y_2 - y_1$ is non-zerodivisor in $\frac{k[x_1,x_2,y_1,y_2]}{J_2}$

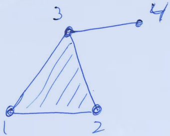
- J_2 is
  - square free
  - from successive separations of I

  SAY: J_2 is polarization of I.

Stanley-Reisner correspondence

- V set. Simplicial complex on V: Family Δ of subsets of V such that: $F \in \Delta$ and $G \subseteq F \implies G \in \Delta$

- $V = \{1,2,3,4\}$, Δ: $\{1,2,3\}, \{3,4\}$
  $\{1,2\}, \{1,3\}, \{2,3\}$
  $\{1\}, \{2\}, \{3\}, \{4\}$
  $\emptyset$

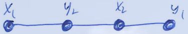
  Δ: 

- $\rightsquigarrow$ Squarefree monomial ideal $I_\Delta = (x_1x_4, x_2x_4)$ generated by monomials $x_F = \prod_{i \in F} x_i$ such that $F \notin \Delta$

- Simplicial complexes on V $\xleftrightarrow{\text{1-1}}$ Squarefree monomial ideals in polynomial ring $k[x_v]_{v \in V}$.

Simple separation

- Example: Simplicial complex associated to $J_2 = (x_1x_2, x_1y_1, y_1y_2)$ is:



- $I \overset{\text{ideal}}{\subseteq} k[x_1, x_2, \dots, x_n]$, $J \overset{\text{ideal}}{\subseteq} k[x_1^1, x_1^2, x_2, x_3, \dots, x_n]$.

Definition: J is a simple separation of I if:

1. I is the image of J by the map $\begin{matrix} x_1^1 \mapsto \\ x_1^2 \mapsto \end{matrix} x_1$

2. Each of x_1^1 and x_1^2 occur in minimal generators of J

3. $x_1^1 - x_1^2$ is a non-zero divisor of $\dfrac{k[x_1^1, x_1^2, x_2, \dots, x_n]}{J}$

Separation, polarization and separated models

J is a *separation* of I it is obtained by a succession of simple separations.

Separation, polarization and separated models

J is a *separation* of I it is obtained by a succession of simple separations.

J is a *separated model* (of I) if it cannot be further separated.

Separation, polarization and separated models

J is a *separation* of I it is obtained by a succession of simple separations.

J is a *separated model* (of I) if it cannot be further separated.

J is a *polarization* of I it is:

- a separation;
- squarefree.

Power of maximal graded ideal

Three variables

$$I = (x,y,z)^2 = (x^2, xy, xz, y^2, yz, z^2) \subseteq k[x,y,z].$$

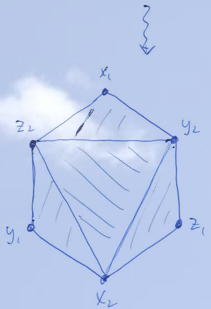
Two distinct polarizations:

$$J_1 = (x_1x_2, x_1y_1, x_1z_1, y_1y_2, y_1z_1, z_1z_2) \subseteq k[x_1,x_2,y_1,y_2,z_1,z_2]$$
$$J_2 = (x_1x_2, x_1y_2, x_1z_2, y_1y_2, y_1z_2, z_1z_2) \subseteq k[x_1,x_2,y_1,y_2,z_1,z_2]$$

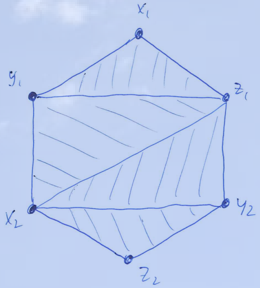
Two polarizations

Three variables

I. $(x_1x_2, x_1y_1, y_1y_2, x_1z_1, y_1z_1, z_1z_2) \subseteq k[x_1, x_2, y_1, y_2, z_1, z_2]$

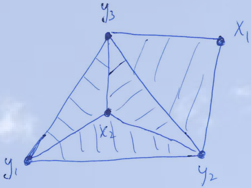


II. $(x_1x_2, x_1y_2, y_1y_2, x_1z_2, y_1z_2, z_1z_2) \subseteq k[x_1, x_2, y_1, y_2, z_1, z_2]$



Polarization

Artinian monomial ideal, two variables

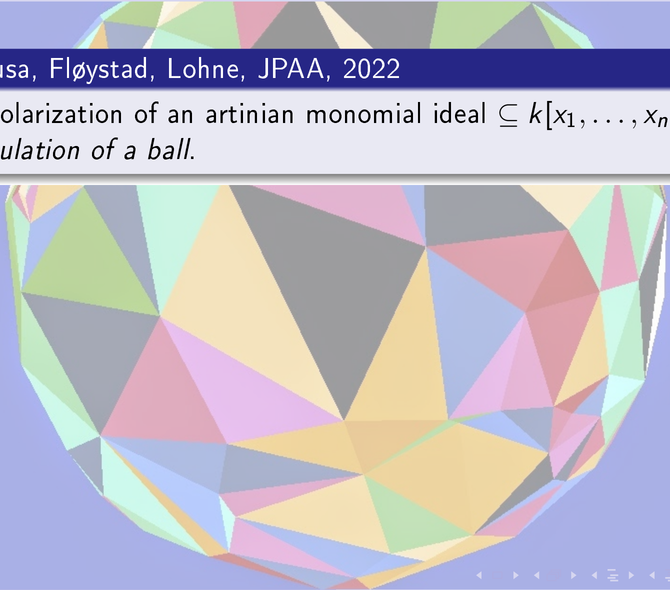
- $I = (x^2, xy, y^3) \subseteq k[x,y]$
- Polarization $J = (x_1 x_2, x_1 y_1, y_1 y_2 y_3) \subseteq k[x_1, x_2, y_1, y_2, y_3]$
- 

Conjecture

Polarizations of Artinian monomial ideals

Almousa, Fløystad, Lohne, JPAA, 2022

Any polarization of an artinian monomial ideal $\subseteq k[x_1, \ldots, x_n]$ is a *triangulation of a ball*.



Conjecture

Polarizations of Artinian monomial ideals

Almousa, Fløystad, Lohne, JPAA, 2022

Any polarization of an artinian monomial ideal $\subseteq k[x_1, \ldots, x_n]$ is a *triangulation of a ball.*

- Shown for standard polarizations (S.Murai, JCSA)
- Shown for $n = 3$ (Almousa, Fløystad, Lohne, JPAA, 2022)
- Shown for letterplace ideals (D'Ali, Fløystad, Nematbakhsh, TAMS, 2019)

Conjecture

Polarizations of Artinian monomial ideals

Almousa, Fløystad, Lohne, JPAA, 2022

Any polarization of an artinian monomial ideal $\subseteq k[x_1, \ldots, x_n]$ is a *triangulation of a ball.*

- Shown for standard polarizations (S.Murai, JCSA)
- Shown for $n = 3$ (Almousa, Fløystad, Lohne, JPAA, 2022)
- Shown for letterplace ideals (D'Ali, Fløystad, Nematbakhsh, TAMS, 2019)

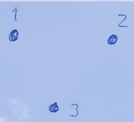
By a result of A.Björner: Enough to show:

- Any polarization of an artinian monomial ideal is shellable, or even weaker:
- Any polarization of an artinian monomial ideal is constructible

Example

Separation and join

- $K = (xz, xy, yz) \subseteq k[x,y,z] = T \rightsquigarrow$ Simplicial Complex



Resolution $\quad K \leftarrow T(-2)^3 \leftarrow T(-3)^2$

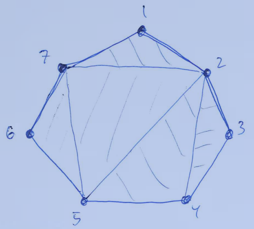
- $I = (x^2, xy, y^2)$

Separation $\nearrow$ $J = (x_1x_2,\ x_1y_1,\ y_1y_2)$

join $\searrow$ $K = (x_1z,\ x_1y_1,\ y_1z)$

$x_2 \mapsto z$, $y_2 \mapsto z$

Triangulation of heptagon



Stanley-Reisner ideal

$$K = (u_1u_3, u_1u_4, \ldots, u_4u_7) \subseteq k[u_1,u_2,u_3,u_4,u_5,u_6,u_7]$$

$$J \subseteq k[x_1,x_2,y_1,y_2,z_1,z_2,w_1,w_2]$$

Separation

Join

$$U \subseteq k[u_1,u_2,u_3,u_4,u_5,u_6,u_7]$$

$$I = (x,y,z,w)^2 \subseteq k[x,y,z,w]$$

Polarizations of second power of graded maximal ideal

Bijection with trees

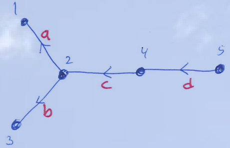
- Polarizations of $(x_1, x_2, \ldots, x_m)^2 \subseteq k[x_1, x_2, \ldots, x_m] \overset{1-1}{\longleftrightarrow}$ Trees with m edges

- Path in oriented tree

$\rightsquigarrow$ monomial $m_{v,w} = x_{e,0} \cdot x_{f,1}$

$\rightsquigarrow$ monomial $m_{v,w} = x_{e,1} \cdot x_{f,1}$

- T :



$\rightsquigarrow$ Ideal $J = I(T) = (m_{ij})_{1 \le i < j \le 5}$

$\subseteq k[x_{a0}, x_{a1}, x_{b0}, x_{b1}, x_{c0}, x_{c1}, x_{d0}, x_{d1}]$

$m_{12} = x_{a1} x_{a0}$ $\quad m_{13} = x_{a1} x_{b1}$ $\quad \ldots$

$m_{14} = x_{a1} x_{c0}$ $\quad \vdots$

Stacked simplicial complexes

Definition (Stacked simplicial complex)

Let X be a pure simplicial complex

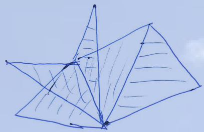
Order facets $F_1, F_2, \ldots, F_m$. Let X_p subcomplex generated by $F_1, F_2, \ldots, F_p$.

If each X_p can be obtained from X_{p-1} by attaching F_p to a *single* codimension one face of X_{p-1}, then X is a *stacked simplicial complex*.

How to get stacked simplicial complexes

Separating and joining

- Example



- Theorem (Fløystad, Orlich 2021)

Any stacked simplicial complex Δ is obtained by:

$J \subseteq k[x_1^1, x_1^2, x_2^1, x_2^2, \dots, x_m^1, x_m^2]$

polarize

join

I_Δ

$I = (x_1, \dots, x_m)^2$

In particular I_Δ and I have same graded Betti numbers

Independent vertex sets

Definition

Let G be a graph with vertex set V. A subset $W \subseteq V$ is an *independent vertex set* if there is no edge with both vertices in W.

Regular subspaces

- T a tree with vertex set V and edge set E.
- By previous construction we get ideal $I(T)$ in $k[x_{E_{01}}]$.

Regular subspaces

- T a tree with vertex set V and edge set E.
- By previous construction we get ideal $I(T)$ in $k[x_{E_{01}}]$.
- $\langle x_{e0}, x_{e1} \rangle_{e \in E}$ be the linear space generated by the variables.

Regular subspaces

- T a tree with vertex set V and edge set E.
- By previous construction we get ideal $I(T)$ in $k[x_{E_{01}}]$.
- $\langle x_{e0}, x_{e1} \rangle_{e \in E}$ be the linear space generated by the variables.
- A subspace $L \subseteq \langle x_{e0}, x_{e1} \rangle_{e \in E}$ with a *basis* of variable differences, is a regular subspace if this basis is a regular sequence for $\frac{k[x_{E_{01}}]}{I(T)}$.

Quotients

Fløystad-Orlich, 2021

Theorem

Regular linear spaces for $k[x_{E_{01}}]/I(T)$
$\overset{1-1}{\longleftrightarrow}$ *partitions of the vertex set of* V.

Quotients

Fløystad-Orlich, 2021

Theorem

Regular linear spaces for $k[x_{E_{01}}]/I(T)$
$\overset{1-1}{\longleftrightarrow}$ *partitions of the vertex set of* V.

Theorem

Regular linear spaces for $k[x_{E_{01}}]/I(T)$ *giving squarefree quotients*
$\overset{1-1}{\longleftrightarrow}$ *partitions of the vertex set of* V *into independent vertex sets*
$\overset{1-1}{\longleftrightarrow}$ *partitions of the edge set* E

Quotients

Fløystad-Orlich, 2021

Theorem

Regular linear spaces for $k[x_{E_{01}}]/I(T)$
$\overset{1-1}{\longleftrightarrow}$ *partitions of the vertex set of* V.

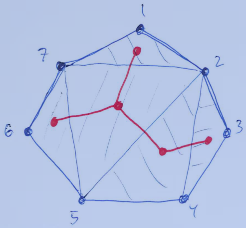
Theorem

Regular linear spaces for $k[x_{E_{01}}]/I(T)$ *giving squarefree quotients*
$\overset{1-1}{\longleftrightarrow}$ *partitions of the vertex set of* V *into independent vertex sets*
$\overset{1-1}{\longleftrightarrow}$ *partitions of the edge set* E

Theorem

Regular linear spaces for $k[x_{E_{01}}]/I(T)$ *giving triangulations of balls*
$\overset{1-1}{\longleftrightarrow}$ *partitions of the edge set* E *into independent vertex sets*

Tree of heptagon



Stanley-Reisner ideal

$\rightsquigarrow$ $K = (u_1u_3, u_1u_4, \dots, u_4u_7) \subseteq k[u_1, u_2, u_3, u_4, u_5, u_6, u_7]$

$J \subseteq k[x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2]$

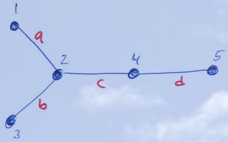
Separation

Join

$U \subseteq k[u_1, u_2, u_3, u_4, u_5, u_6, u_7]$

$I = (x, y, z, w)^2 \subseteq k[x, y, z, w]$

Partitions

- T: 

- We get ideal $\mathcal{V}$ of heptagon by
  - joining variables in $I(T)$
  - dividing by regular variable difference

- Partition of vertices $\{1,5\}, \{2\}, \{3\}, \{4\}$. Partition of edges $\{a,d\}, \{b\}, \{c\}$.

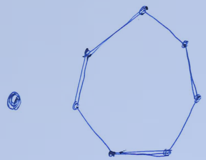
- $m_{15} = x_{a1} x_{d0} \rightsquigarrow$ Divide out by $x_{a0} - x_{d1}$.

Separate and join variables

Underused

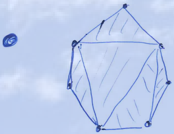
- X a simplicial complex
- Can its Stanley-Reisner ideal be:
  - separated
  - joined?

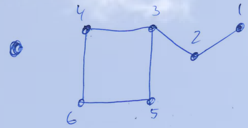
Examples



Cannot be separated
(homology is obstruction)

To separate, fill in

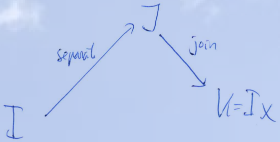




Can separate $X_3 \to X_3^1$, $X_3 \to X_3^2$

Structural insight

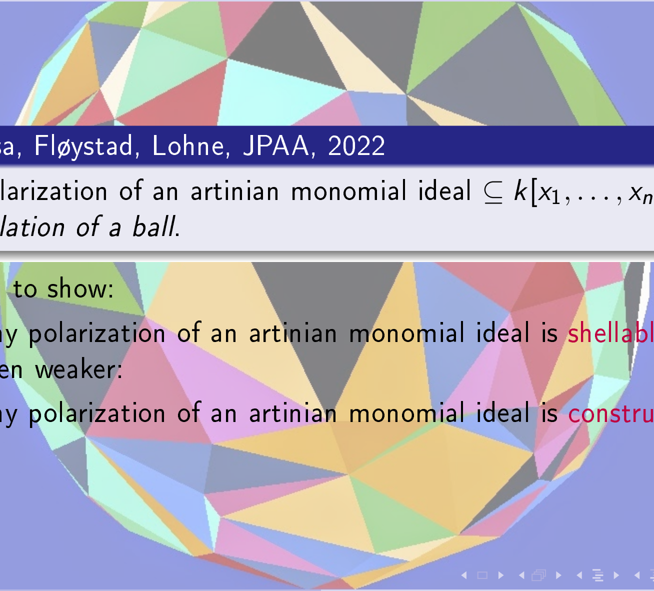
- X contractible simplicial complex, for instance triangulation of ball.
- Can you separate $V = I_X$? Good chance ⇝ Separated model J.
- Is J a polarization of an artinian monomial ideal I?
- Get



structural insight.

Conjecture

Polarizations of Artinian monomial ideals



Almousa, Fløystad, Lohne, JPAA, 2022

Any polarization of an artinian monomial ideal $\subseteq k[x_1, \ldots, x_n]$ is a *triangulation of a ball.*

Enough to show:

- Any polarization of an artinian monomial ideal is shellable, or even weaker:
- Any polarization of an artinian monomial ideal is constructible

Stockholm