NOTES ABOUT HODGE THEORY

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Contents

1. Introduction

The aim of this short note is to give some preliminary ideas of what we are supposed to know before starting with serious Hodge theory and briefly discuss the serious aspects of the theory. In addition, some of the material presented here will contain explanations and formalization of concepts that – if included in the lectures – will obscure the presentation, making us loose perspective of what we want to achieve.

As you know, I will be basing these lectures on Chapter 6 of the book

- F. Warner, "Foundations of differentiable manifolds and Lie groups" (GTM-94), and in case I need some extra material, I will be looking at

- S. Rosenberg, "The Laplacian on a Riemannian manifold" (LMS student texts 31),
- R. Palais, "Seminar on the Atiyah-Singer index theorem" (AM-57) or
- R. Bott and L. Tu, "Differential forms in algebraic topology" (GTM-82).

Warning: As many of you can imagine, the hardest part of starting at the last chapter of a book, is keeping track of the prerequisites. I will try to do my best.

1.1. **What should we expect from the seminar.** Hodge theory is a set of analytical tools proposed by William V. D. Hodge around 1930 to study the geometry of compact Riemannian and Kähler manifolds. In these lectures we will focus on the Riemannian world, since to my knowledge this is the place where the most analysis is used. There is a reformulation of Hodge theory, by Dolbeault, in the case of compact Kähler manifolds that uses not-that-much analysis and it is mostly algebro-geometric in spirit[1]

The basic idea is to define correctly what is the (Hodge) Laplacian Δ of $k-$forms on a compact, connected, oriented Riemannian manifold M of dimension n. Here we have (to my knowledge) two paths to follow: one using elliptic techniques (as in Warner) and the other using parabolic techniques (as in Rosenberg). Since we will be following the first path, let me make a small digression concerning the second one.

The definition of Hodge Laplacian, permits us to pose the problem of finding the fundamental solution $e(t,x,y)$ to the heat equation

$$(\partial_t + \Delta_x)e(t,x,y) = 0, \tag{1}$$

where e is a section of the bundle $\mathbb{R}^+ \times \Omega^k(M) \otimes \Omega^k(M)$.[2] Analytic theorems, such as Rellich-Kondrachov compactness and Sobolev embedding, will then be used to show that the associated heat operator $e^{-t\Delta} : L_2(\Omega^k(M)) \to L_2(\Omega^k(M))$ defined by

$$(e^{-t\Delta} f)(x) = \int_M e(t,x,y) f(y) d\text{vol}(y) \tag{2}$$

is compact and self-adjoint. This fact will have as a consequence that there is an orthonormal basis of $L^2(\Omega^k(M))$ consisting of eigenforms of Δ, where each eigenvalue is nonnegative, has finite multiplicity and the sequence of

[1]For the inspired minds, you can consult C. Voisin, "Hodge theory and complex algebraic geometry I & II", Cambridge studies in advanced mathematics 76 & 77.

[2]For those not familiar with the terminology, just keep reading. This is intended as a bad introduction: is an introduction for those who know what we will do.

eigenvalues accumulates at infinity. From this characterization, we can obtain some really neat theorems:

- (Convergence to the mean) $\lim_{t\to 0}(e^{-t\Delta}f)(x) = \dfrac{1}{\mathrm{vol}(M)}\displaystyle\int_M f\,d\mathrm{vol}$, for all $x \in M$ (*i.e.* the heat flow sends a function to its average as $t \to \infty$).
- (Hodge decomposition theorem) $\Omega^k(M) = \ker\Delta \oplus \operatorname{im} d \oplus \operatorname{im}\delta$, where d is the differential, δ is the codifferential and the decomposition is orthogonal. This result relies strongly on regularity results.
- The canonical map $\ker\Delta^k \to H^k_{dR}(M)$ given by $\omega \mapsto [\omega]$ is an isomorphism ($H^k_{dR}(M)$ stands for the kth de Rham cohomology group).
- (Smooth Poincaré duality) $H^k_{dR}(M) \cong H^{n-k}_{dR}(M)$.
- (Maximum principle) A harmonic function on M has to be constant.
- The Euler characteristic of an odd dimensional compact manifold is zero.

The approach that we will follow gives as a consequence all but the first of these results. If time permits, we will see (besides the above coolness) a characterization of the eigenvalues of the Laplacian and a very nice proof of the weak Peter–Weyl theorem (which says that the representative ring of a Lie group is dense in the space of complex valued continuous functions with respect to the uniform norm). See Exercises 6.16 and 6.20 in Warner.

1.2. **What I will definitely assume everyone knows.** I will (of course) need a good deal of familiarity with manifolds. Some knowledge of Fourier analysis techniques and PDEs will be appreciated, but not strictly necessary. As usual with my seminars, I will be recalling theorems from linear algebra and classical calculus (in several variables) as much as I can. I will try to avoid notions that are "too algebraic" like sheaves, exact sequences or stuff like that. I will write down some definitions and examples of those things in case it is strictly necessary. It is of course convenient to know them, but it will not be a fundamental part of the seminars.

2. A little linear algebra

As the title of the section states, the main idea is to give a very brief summary of the things you should bare in mind. Proofs and discussions can be found in Warner (Chapters 2 and 4).

2.1. Exterior algebra bundle and differential forms.

Definition 1. *The $k-$fold exterior product of V is a vector space $\Lambda^k(V)$, together with a linear map*

$$\theta : V^k = \underbrace{V \times \cdots \times V}_{k\ \text{times}} \to \Lambda^k(V)$$

determined by the following universal property: If $\varphi : V^k \to W$ is an alternating multilinear[3] *map (for some vector space W), then there is a* **unique** *map $\psi : \Lambda^k(V) \to W$ such that $\psi \circ \theta = \varphi$.*

The *exterior algebra* $\Lambda(V) = \bigoplus_k \Lambda^k(V)$ is a graded algebra, with product given by the wedge $\wedge$. For finite dimensional vector spaces (those that we are interested in), it is possible to find an explicit basis for each $\Lambda^k(V)$: If $e_1, \ldots, e_n$ is a basis of V, then the set

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n} \tag{3}$$

is a basis of $\Lambda^k(V)$.
Remark: The existence of the basis (3) has as consequence that whenever $1 \le k \le n$, we have $\dim \Lambda^k(V) = \binom{n}{k}$. In all other cases, $\Lambda^k(V) = 0$. In particular, the set

$$\{\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n}\}_{1 \le k \le n} \tag{4}$$

is a basis of $\Lambda(V)$.

Definition 2. *The kth exterior bundle over M (smooth manifold) is the vector bundle $\Lambda^k(M) = \coprod_{x \in M} \Lambda^k(T^*_x M)$.*[4]

A section of the bundle $\Lambda^k(M) \to M$ is called a differential $k-$form. The set of differential $k-$forms is denoted by $\Omega^k(M)$, and the set of differential forms $\bigoplus_k \Omega^k(M)$ is denoted by $\Omega(M)$. Recall that $\Omega(M)$ has the structure of a module over the ring of smooth functions, and of a graded algebra with wedge multiplication.

[3] A multilinear map $\varphi : V^k \to W$ is alternating if $\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (-1)^{\mathrm{sgn}(\sigma)} \varphi(v_1, \ldots, v_k)$ for all $\sigma \in S_k$, is a permutation of k elements.

[4] $\coprod$ denotes disjoint union.

Definition 3. *The differential in degree k is the map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined locally over monomials by*

$$d(f_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \left(\sum_{r=1}^{n} \frac{\partial f_{i_1,\ldots,i_k}}{\partial x_r} dx_r \right) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

where $x = (x_1, \ldots, x_n)$ denotes coordinates in a chart, and then extended by linearity.

Remark: $d : \Omega(M) \to \Omega(M)$ is the unique map of degree one such that $d^2 = 0$ and $d|_{\Omega^0(M)}$ is the usual differential. See Theorem 2.20 in Warner.

2.2. **Solving exercise 2.13 in Warner.** I have problems believing some of you will even look at an exercise if I tell you to solve it. In Spanish we say that "the devil knows more for being old than by being the devil", so my trust problems are reflected here. I intend to sketch a solution to exercise 2.13 (pp. 79–80) in Warner.

<u>Statement:</u>. Let V be an $n-$dimensional real inner product space. We extend the inner product from V to all of $\Lambda(V)$ by setting the inner product of elements which are homogeneous of different degrees equal to zero, and by setting

$$\langle w_1 \wedge \ldots \wedge w_p, v_1 \wedge \ldots \wedge v_p \rangle = \det \langle w_i, v_j \rangle$$

and then extending bilinearly to all of $\Lambda^p(V)$. Prove that if $e_1, \ldots, e_n$ is an orthonormal basis of V, then the corresponding basis (4) of $\Lambda(V)$ is an orthonormal basis for $\Lambda(V)$.

Since $\Lambda^n(V)$ is one dimensional, $\Lambda^n(V) - \{0\}$ has two components. An *orientation on* V is a choice of a component of $\Lambda^n(V) - \{0\}$. If V is an oriented inner product space, then there is a linear transformation

$$* : \Lambda(V) \to \Lambda(V)$$

called *star*, which is well-defined by the requirement that for *any* orthonormal basis $e_1, \ldots, e_n$ of V (in particular, for any re-ordering of a given basis),

$$*(1) = \pm e_1 \wedge \ldots \wedge e_n, \qquad *(e_1 \wedge \ldots \wedge e_n) = \pm 1,$$
$$*(e_1 \wedge \ldots \wedge e_p) = \pm e_{p+1} \wedge \ldots \wedge e_n,$$

where one takes "+" if $e_1 \wedge \ldots \wedge e_n$ lies in the component of $\Lambda^n(V) - \{0\}$ determined by the orientation and "–" otherwise. Observe that

$$* : \Lambda^p(V) \to \Lambda^{n-p}(V).$$

Prove that on $\Lambda^p(V)$

$$** = (-1)^{p(n-p)}.$$

Also prove that for arbitrary $v, w \in \Lambda^p(V)$, their inner product is given by

$$\langle v, w \rangle = *(w \wedge *v) = *(v \wedge *w).$$

Solution:. First we need to show that if $e_1, \dots, e_n$ is an orthonormal basis of V, then (4) is an orthonormal basis for $\Lambda(V)$. To do this, note that we just need to prove that

(5)

$$\langle e_{i_1} \wedge \ldots \wedge e_{i_p}, e_{j_1} \wedge \ldots \wedge e_{j_p} \rangle = \delta_{i_1,j_1} \cdot \ldots \cdot \delta_{i_p,j_p} = \begin{cases} 1 & , \text{ if } i_1 = j_1, \dots, i_p = j_p \\ 0 & , \text{ else} \end{cases},$$

where $\delta_{i,j}$ is the Kronecker symbol.[5]

It is easy to see that the matrix $(\langle e_{i_k}, e_{i_l} \rangle)$ is the identity matrix[6], thus

$$\langle e_{i_1} \wedge \ldots \wedge e_{i_p}, e_{i_1} \wedge \ldots \wedge e_{i_p} \rangle = \det(\text{Identity matrix of size } p \times p) = 1.$$

On the other hand, if $e_{i_1} \wedge \ldots \wedge e_{i_p} \neq e_{j_1} \wedge \ldots \wedge e_{j_p}$, that means there is a k such that $i_k \neq j_k$[7]. This means, the kth row (or column) of the matrix $(\langle e_{i_k}, e_{i_l} \rangle)$ is zero. This completes the proof of equation (5).

Now we need to show that $** = (-1)^{p(n-p)}$ in $\Lambda^p(V)$. It is of course enough to check it on monomials $e_{i_1} \wedge \ldots \wedge e_{i_p}$[8]. We can assume, without loss of generality, that $*1 = e_1 \wedge \ldots \wedge e_n$. Note that the definition of $*$ implies that[9]

$$e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \ldots \wedge e_{i_p}) = e_1 \wedge \ldots \wedge e_n.$$

Now everything is easy, because since

$$e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \underbrace{*(e_{i_1} \wedge \ldots \wedge e_{i_p})}_{(i)} \wedge \underbrace{**(e_{i_1} \wedge \ldots \wedge e_{i_p})}_{(ii)},$$

then all we have to do is to move (i), $n-p$ slots (because (i) is in $\Lambda^{n-p}(V)$). So we get a $(-1)^{n-p}$ for each of the p basis elements that form (ii) (because (ii) is in $\Lambda^p(V)$). Since this is true for any oriented basis, we have that $** = (-1)^{p(n-p)}$.

Finally, we need to show that $\langle v, w \rangle = *(w \wedge *v) = *(v \wedge *w)$. Again, we can simply work with monomials, and after the previous parts of the exercise, we see that it is very easy. This because

$$*(e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \ldots \wedge e_{i_p})) = *(e_1 \wedge \ldots \wedge e_n) = 1 = \langle e_{i_1} \wedge \ldots \wedge e_{i_p}, e_{i_1} \wedge \ldots \wedge e_{i_p} \rangle,$$

and on the other hand, if there is a k such that $i_k \neq j_k$, then $*(e_{j_1} \wedge \ldots \wedge e_{j_p})$ will be of the form $\pm e_{i_k} \wedge (\text{something})$, which means that

$$*(e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \ldots \wedge e_{i_p})) = 0 = \langle e_{i_1} \wedge \ldots \wedge e_{i_p}, e_{i_1} \wedge \ldots \wedge e_{i_p} \rangle,$$

as claimed.

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[5]Assuming, as usual, that $1 \leq i_1 < \cdots < i_p \leq n$, and that $1 \leq j_1 < \cdots < j_p \leq n$.

[6]Remember that the basis is ordered!

[7]See footnotes 5 and 6.

[8]See footnote 5.

[9]Just by reordering and ordering again.

3. Crash course in integration (on manifolds)

3.1. **Orientation.** Let us start discussing a well known equivalence. The proof is in Warner, but let us shortly review it here for the sake of completeness.

Proposition 1. *The following are equivalent for a smooth manifold M of dimension n:*

(1) $\Lambda^n(M) - O$ *has two connected components, where O denotes the zero section of the bundle* $\Lambda^k(M)$[10],

(2) *There is a collection $\Phi = \{(V, \psi)\}$ of coordinate systems on M such that*

$$M = \bigcup_{(V,\psi)\in\Phi} V \quad \textit{and} \quad \det\left(\frac{\partial x_i}{\partial y_j}\right) > 0 \quad \textit{on } U \cap V,$$

whenever $(U, x_1, \ldots, x_n), (V, y_1, \ldots, y_n) \in \Phi$.

(3) *There is a nowhere-vanishing $n-$form on M.*

Proof. Let us proceed as follows: (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1)**:** This is the easiest one (I think). Let ω be a nowhere-vanishing $n-$form in M, and let

$$\Lambda^+ = \coprod_{x\in M}\{a\omega(x) : a \in \mathbb{R}, a > 0\} \quad , \quad \Lambda^- = \coprod_{x\in M}\{a\omega(x) : a \in \mathbb{R}, a < 0\}.$$

We have $\Lambda^n(M) - O = \Lambda^+ \coprod \Lambda^-$.

(1) $\Rightarrow$ (2)**:** This is the tricky one. Choose one of the components in $\Lambda^n(M) - O$ and call it Λ. The collection Φ consists of coordinate systems $(V, y_1, \ldots, y_n)$ on M such that the map

$$V \ni x \mapsto (dy_1 \wedge \ldots \wedge dy_n)(x) \in \Lambda^n(T_xM) - O$$

has range in Λ. Note that if $(V, y_1, \ldots, y_n) \notin \Phi$, then $(V, y_2, y_1, \ldots, y_n) \in \Phi$. Moreover, since for any two coordinate systems, $(U, z_1, \ldots, z_n)$ and $(V, y_1, \ldots, y_n)$, we have

$$(dy_1 \wedge \ldots \wedge dy_n)(x) = \det\left(\frac{\partial y_i}{\partial z_j}\right)(x)(dz_1 \wedge \ldots \wedge dz_n)(x),$$

for any $x \in U \cap V$, it follows that if $(U, x_1, \ldots, x_n), (V, y_1, \ldots, y_n) \in \Phi$ then $\det\left(\frac{\partial y_i}{\partial z_j}\right)(x) > 0$.

[10] If $\pi : E \to M$ is a vector bundle, then the zero section O of E is the subbundle $\coprod_{x\in M}\{0 \in \pi^{-1}(x)\}$.

(2) $\Rightarrow$ (3): This one is also easy. Let $\{\phi_i\}$ be a partition of unity subordinate to the cover $\{V : (V, \psi) \in \Phi\}$, where ϕ_i is subordinate to $(V_i, x_1^i, \ldots, x_n^i)$. Then

$$\omega = \sum_i \phi_i \, dx_1^i \wedge \ldots \wedge dx_n^i$$

will do.

□

3.2. **Integration over chains.** Let $p \geq 1$. The standard $p-$simplex in $\mathbb{R}^p$ is the set

$$\Delta^p = \left\{ (a_1, \ldots, a_p) \in \mathbb{R}^p : \sum_{i=1}^{p} a_i \leq 1 \right\}.$$

As convention $\Delta^0 = \{0\}$ the one-point space[11]. A differentiable singular $p-$simplex σ in M is a map $\sigma : \Delta^p \to M$ which extends to a differentiable map of a neighborhood of Δ^p in $\mathbb{R}^p$ into M. For the ease of notation, we will just refer to differentiable singular $p-$simplices as $p-$simplices.

Definition 4. *A $p-$chain c in M is a (formal) finite linear combination $c = \sum a_i \sigma_i$ of $p-$simplices σ_i in M, where the a_is are real numbers. The set of $p-$chains is denoted by $C^p(M)$.*

The ith face of a $p-$simplex σ is the $(p-1)-$simplex σ^i defined by $\sigma^i = \sigma \circ k_i^{p-1}$, where the maps $k_i^p : \Delta^p \to \Delta^{p+1}$, $0 \leq i \leq p+1$, are defined by

$$\begin{cases} \text{for} \quad p = 0, \quad k_0^0(0) = 1 \quad \text{and} \quad k_1^0(0) = 0 \\ \text{for} \quad p \geq 1, \quad \begin{cases} k_0^p(a_1, \ldots, a_p) = \left(1 - \displaystyle\sum_{i=1}^{p} a_i, a_1, \ldots, a_p\right) \quad \text{and} \\ k_i^p(a_1, \ldots, a_p) = (a_1, \ldots, a_{i_1}, 0, a_i, \ldots, a_p) \quad 1 \leq i \leq p+1. \end{cases} \end{cases}$$

The boundary $\partial\sigma$ of a $p-$simplex σ is the $(p-1)-$chain

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma^i.$$

Extending it linearly we get an operator $\partial : C^p(M) \to C^{p-1}(M)$.

[11] I'm sorry about the notation. The Laplacian on $p-$forms will be denoted in the same way. I prefer to keep Warner's notation and not to be confused.

Fact:. $\partial \circ \partial = 0$. This follows from the fact that $k_i^{p+1} \circ k_j^p = k_{j+1}^{p+1} \circ k_i^p$, $p \geq 0$, $i \leq j$ (see Warner, pp. 143–144).

Definition 5. *The integral of a $0-$form ω (a function) over a $0-$simplex σ (a point $\sigma(0)$) is defined as*

$$\int_\sigma \omega = \omega(\sigma(0)).$$

The integral of a $p-$form ω over a $p-$simplex σ is defined as

$$\int_\sigma \omega = \int_{\Delta^p} \delta\sigma(\omega),$$

where $\delta\sigma(\omega)$ is the $p-$form obtained by pulling-back ω to (a neighborhood of) Δ^p via σ. Extending this definition linearly we get a "pairing"

$$C^p(M) \times \Omega^p(M) \xrightarrow{\int} \mathbb{R}, \qquad \left(\sum_i a_i\sigma_i, \omega\right) \mapsto \int_c \omega = \sum_i a_i \int_{\sigma_i} \omega.$$

One of the most important theorems when integrating over chains is Stokes(-Ostrogradskiĭ)' formula. It generalizes in a very straightforward manner the classical fundamental theorem of calculus.

Theorem 1 (Stokes'). *Let $p \geq 1$, c be a $p-$chain and ω a $(p-1)-$form defined on a neighborhood of the image of c. Then*

$$\int_{\partial c} \omega = \int_c d\omega$$

Proof. See Warner, pp. 144–145, or Spivak's "Calculus on Manifolds", pp. 94–96 (in my edition), or wherever you find it. It is not hard, but it's lengthy to write. □

3.3. **Integration on an oriented manifold.** for the sake of simplicity (and because the general case is simply more involved technically, but not mathematically harder to understand), let us study how to define the integral on the whole manifold[12], following Bott and Tu's "Differential forms in algebraic topology".

Before starting, let us denote by $\Omega_c^p(M)$ the space of compactly supported $p-$differential forms. It consists of $p-$forms whose coefficients are compactly supported functions[13] in M.

[12] In Warner there's a more general definition of the integral over a regular domain. The interested reader can check pp. 145–148.

[13] In Bott and Tu they define the algebra Ω^* as the real algebra generated by $dx_1, \ldots, dx_n$ with the relations

$$\begin{cases} dx_i \wedge dx_i = 0 \\ dx_i \wedge dx_j = -dx_j \wedge dx_i. \end{cases}$$

In this language, $\Omega_c^p(M) = C_c^\infty(M) \otimes_{\mathbb{R}} \Omega^*$.

Let us choose an orientation of M and denote it by $[M]$. Let $\tau \in \Omega_c^n(M)$. We define its integral by

$$\int_{[M]} \tau = \sum_i \int_{U_i} \rho_i \tau = \sum_i \int_{\mathbb{R}^n} (\varphi_i^{-1})^*(\rho_i \tau)$$

where $\{(U_i, \varphi_i)\}$ is an oriented atlas and $\{\rho_i\}$ a partition of unity subordinate to the covering $\{U_i\}$. Understanding the orientation (and reversing signs if necessary), we can simply write $\int_M \tau$.

Theorem 2. *The definition of* $\int_M \tau$ *given above does not depend on the choice of* $\{(U_i, \varphi_i)\}$ *or* $\{\rho_i\}$.

Proof. Let $\{(V_j, \psi_j)\}$ be another oriented atlas and $\{\chi_j\}$ a partition of unity subordinate to the covering $\{V_j\}$. Since

$$\sum_i \int_{U_i} \rho_i \, \tau = \sum_{i,j} \int_{U_i} \rho_i \, \chi_j \, \tau$$

(because $\{V_j\}$ is a partition of unity) and

$$\int_{U_i} \rho_i \, \chi_j \, \tau = \int_{V_j} \rho_i \, \chi_j \, \tau$$

(because the support of $\rho_i \, \chi_j \, \tau$ is in $U_i \cap V_j$), we have

$$\int_M \tau = \sum_i \int_{U_i} \rho_i \, \tau = \sum_{i,j} \int_{U_i} \rho_i \, \chi_j \, \tau = \sum_{i,j} \int_{V_j} \rho_i \, \chi_j \, \tau = \sum_j \int_{V_j} \chi_j \, \tau.$$

□

In order to state Stokes' theorem in this context we need to walk around a little in the realm of manifolds with boundary. It won't be really a big deal, but it might be good to take a look at this sometime.

Definition 6. *We say that* M *is a manifold with boundary if there is an atlas* $\{(U_i, \varphi_i)\}$ *such that each* U_i *is homeomorphic either to* $\mathbb{R}^n$ *or to the upper half space* $\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_n \geq 0\}$. *The boundary* ∂M *of* M *is an* $(n-1)-$*dimensional manifold.*

An oriented atlas of M induces an oriented atlas of ∂M. In order to see this, we first need the following Lemma.

Lemma 1. *Let* $T : \mathbb{H}^n \to \mathbb{H}^n$ *be a diffeomorphism with everywhere positive Jacobian. Then* T *induces a map* $\bar{T} : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ *which, as a diffeomorphism of* $\mathbb{R}^{n-1}$ *to itself, has positive Jacobian.*

Proof. By the inverse function theorem, the preimage of an interior point of $\mathbb{H}^n$ is an interior point of $\mathbb{H}^n$. This implies the map $\bar{T}$ is well defined.

To see that it has a positive Jacobian, let us first write $x_i = T_i(y_1, \ldots, y_n)$, for $1 \le i \le n$. Observe that $\bar{T}$ is given by

$$\bar{T}(y_1, \ldots, y_{n-1}) = (T_1(y_1, \ldots, y_{n-1}, 0), \ldots, T_{n-1}(y_1, \ldots, y_{n-1}, 0)).$$

By hypothesis, we know that

$$\det \left(\begin{array}{c|c} \left(\dfrac{\partial T_i}{\partial y_j}(y_1, \ldots, y_{n-1}, 0)\right)_{1 \le i,j \le n-1} & \left(\dfrac{\partial T_i}{\partial y_n}(y_1, \ldots, y_{n-1}, 0)\right)_{1 \le i \le n-1} \\ \hline \left(\dfrac{\partial T_n}{\partial y_j}(y_1, \ldots, y_{n-1}, 0)\right)_{1 \le j \le n-1} & \dfrac{\partial T_n}{\partial y_n}(y_1, \ldots, y_{n-1}, 0) \end{array} \right) > 0.$$

It is clear that $T_n(y_1, \ldots, y_{n-1}, 0) = 0$ and also that[14] $\dfrac{\partial T_n}{\partial y_n}(y_1, \ldots, y_{n-1}, 0) > 0$. It follows that

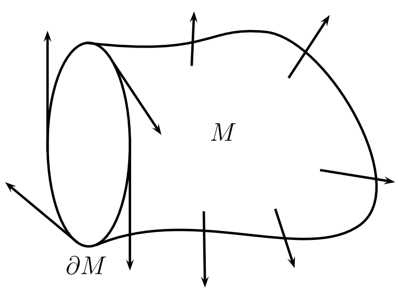
$$\det \left(\frac{\partial T_i}{\partial y_j}(y_1, \ldots, y_{n-1}, 0)\right)_{1 \le i,j \le n-1} = \text{Jacobian of } \bar{T} > 0.$$

$\square$

We give to the upper-half space $\mathbb{H}^n = \{x_n \ge 0\}$ in $\mathbb{R}^n$ the standard orientation $dx_1 \wedge \ldots \wedge dx_n$. The induced orientation on its boundary is the equivalence class of

$$(-1)^n dx_1 \wedge \ldots \wedge dx_{n-1}$$

for $n \ge 2$ and -1 for $n = 1$. On a general manifold with boundary, we induce an orientation of the boundary by pulling-back the orientation of $\partial \mathbb{H}^n$ via local diffeomorphisms that are orientation preserving in the interior. Note that this definition agrees with the "engineering" one given by the right hand rule (see the figure below).



Theorem 3. *If M is oriented, its boundary ∂M has the induced orientation and $\omega \in \Omega_c^{n-1}(M)$, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

[14] Recall that T maps the upper-half space to itself.

Proof. See Spivak, Warner, Bott–Tu, Arnol'd, Lang, etc. Not really hard, but you have to be careful. As the great wise Erlend once said "everybody knows Stokes' theorem". □

Corollary 1. *Let* $\omega \in \Omega^{n-1}(M)$ *and* M *compact oriented and without border, then*

$$\int_M d\omega = 0.$$

3.4. **Introduction to de Rham cohomology.** A sequence of vector spaces and maps (V_i, d_i), where $d_i : V_i \to V_{i+1}$, is called a complex if $d_{i+1} \circ d_i = 0$. The example we are interested in is the complex

$$\cdots \xrightarrow{d} \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots$$

The $p-$th de Rham cohomology of M is the vector space

$$H^p_{dR}(M) = \frac{\ker\{d : \Omega^p(M) \to \Omega^{p+1}(M)\}}{\operatorname{Im}\{d : \Omega^{p-1}(M) \to \Omega^p(M)\}}.$$

Elements in $\ker\{d : \Omega^p(M) \to \Omega^{p+1}(M)\}$ are usually called closed $p-$forms and elements in $\operatorname{Im}\{d : \Omega^{p-1}(M) \to \Omega^p(M)\}$ are usually called exact $p-$forms. More explicitly, a $p-$form $\alpha \in \Omega^p(M)$ is closed if $d\alpha = 0$ and it is exact if there is $\beta \in \Omega^{p-1}(M)$ such that $d\beta = \alpha$. Since $d \circ d = 0$, it is clear that every exact form is closed, which explains why we can take the quotient in the definition of de Rham cohomology.

Note that if M is compact oriented and without border, then the map

$$\Omega^n(M) \ni \omega \mapsto \int_M \omega$$

descends to a well-defined map $H^n_{dR}(M) \to \mathbb{R}$ (because, by Stokes' theorem, the integral of an exact form is zero).

Finally observe that if $f : M \to N$ is a smooth map, then we have an induced homomorphism (linear map)

$$f^* : H^p_{dR}(N) \to H^p_{dR}(M),$$

for each integer $p \geq 0$. Moreover, we see that

$$(g \circ f)^* = f^* \circ g^*$$

and $(\mathrm{id}_M)^* = \mathrm{id}_{H^p_{dR}(M)}$. It follows that if $f : M \to N$ is a diffeomorphism, then f^* is an isomorphism of the vector spaces $H^p_{dR}(N)$ and $H^p_{dR}(M)$.

4. Volume form and the $*-$operator

4.1. **The volume form in vector spaces.** Let V be an oriented inner product vector space, with inner product $\langle\cdot,\cdot\rangle$, of dimension $n \geq 1$. The volume form[15] on V is the unique $\omega \in \Lambda^n(V^*)$ such that for any positively oriented basis $v_1, \dots, v_n$ we have

$$\omega = \sqrt{\det(\langle v_i, v_j\rangle)}\, v_1^* \wedge \ldots \wedge v_n^*,$$

where $v_1^*, \dots, v_n^*$ denotes the corresponding dual basis. Note that if $w_1, \dots, w_n$ is another positively oriented basis, i.e. $w_i = \sum_{k=1}^n a_{ik} v_k$ and $A = (a_{ij})$ has positive determinant, then

$$\begin{aligned}
\sqrt{\det(\langle w_i, w_j\rangle)}\, w_1^* \wedge \ldots \wedge w_n^* &= \sqrt{\det(\langle w_i, w_j\rangle)}\det(A^{-1})\, v_1^* \wedge \ldots \wedge v_n^* \\
&= \sqrt{\det(A\langle v_i, v_j\rangle A^T)}(\det A)^{-1}\, v_1^* \wedge \ldots \wedge v_n^* \\
&= \sqrt{\det(\langle v_i, v_j\rangle)}\, v_1^* \wedge \ldots \wedge v_n^*.
\end{aligned}$$

In other words, the form ω does not depend on the chosen basis, as long as it is positively oriented.

4.2. **The volume form in manifolds.** Now, let us consider (M, g) be an oriented Riemannian manifold of dimension n. Now consider a positively oriented coordinate system $(V, y_1, \dots, y_n)$ of M, then the following

$$\mathrm{dvol} = \sqrt{\det g}\, dy_1 \wedge \ldots \wedge dy_n$$

gives a globally defined $n-$form. Note that the integral

$$\int_M \mathrm{dvol}$$

is positive (and possibly infinite). This is called the volume of M and it is usually denoted by $\mathrm{vol}(M)$. Observe that by Stokes' theorem, the notation for the volume form is deceitful, because it is not an exact form.

4.3. **The Hodge $*-$operator.** Note that the $*-$operator introduced in Subsection 2.2 extends naturally to a linear map

$$* : \Omega^p(M) \to \Omega^{n-p}(M).$$

This operator is called the Hodge $*-$operator. Note that[16] $*1 = \mathrm{dvol}$.

Let us shortly recall that for each $x \in M$, and any positively oriented basis $\theta_1, \dots, \theta_n$ of T_x^*M, the map $*_x : \Lambda^p(T_x^*M) \to \Lambda^{n-p}(T_x^*M)$ is defined by

$$*_x(\theta_{i_1} \wedge \ldots \wedge \theta_{i_p}) = \theta_{j_1} \wedge \ldots \wedge \theta_{j_{n-p}},$$

[15] It depends on the orientation and on the chosen inner product.

[16] Here 1 denotes the constant function $M \ni x \mapsto 1$, which is of course a $0-$form.

where $i_1 < \ldots < i_p$ and $\theta_{i_1} \wedge \ldots \wedge \theta_{i_p} \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_{n-p}} = \theta_1 \wedge \ldots \wedge \theta_n$. Equivalently, $*_x$ defined abstractly as follows. For each $x \in M$ the right exterior product induces a natural[17] isomorphism

$$p : \Lambda^{n-p}(T_x^*M) \to \mathrm{Hom}(\Lambda^p(T_x^*M), \Lambda^n(T_x^*M)).$$

The volume form on M gives a canonical identification $\Lambda^n(T_x^*M) \cong \mathbb{R}$, and moreover the metric induces the isomorphism[18]

$$m : \Lambda^p(T_x^*M) \to \mathrm{Hom}(\Lambda^p(T_x^*M), \mathbb{R}).$$

The (pointwise) Hodge operator[19] is defined as $*_x = p^{-1} \circ m$.

From now on, let us assume M is compact and oriented. Observe that by Subsection 2.2, it follows that $** = (-1)^{p(n-p)}$ and also that

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

defines an inner product in $\Omega^p(M)$. As usual, we denote by $\| \cdot \|$ the corresponding norm (i.e. $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$). It extends to an inner product on $\Omega(M)$ by assuming $\Omega^p(M)$ and $\Omega^q(M)$ are orthogonal subspaces, for $p \neq q$. For a $0-$form f (a function $f : M \to \mathbb{R}$), we define the integral of f over M as the integral of $*f$, i.e.

$$\int_M f = \int_M *f = \int_M f \,\mathrm{dvol}.$$

Proposition 2. *If $\alpha \in \Omega^{p+1}(M)$ and $\beta \in \Omega^p(M)$, then*

$$\langle d\beta, \alpha \rangle = (-1)^{np+1} \langle \beta, *d * \alpha \rangle.$$

[17]Meaning independent of the choice of a basis.

[18]Remember that if V is a real vector space, then $\mathrm{Hom}(V, \mathbb{R})$ is nothing but the dual space. What this is saying is that if V is a finite dimensional inner product space, the isomorphism between V and its dual is natural.

[19]Is it clear that both definitions are equivalent?

Proof. We have the following chain of equalities

$$\begin{array}{rcl}
\langle d\beta, \alpha \rangle & = & \int_M d\beta \wedge *\alpha \\
& \overset{(1)}{=} & \int_M d(\beta \wedge *\alpha) - (-1)^p \int_M \beta \wedge d * \alpha \\
& \overset{Stokes'}{=} & (-1)^{p+1} \int_M \beta \wedge d * \alpha \\
& = & (-1)^{p+1}(-1)^{(n-p)(n-(n-p))} \int_M \beta \wedge * * d * \alpha \\
& = & (-1)^{np-p(p-1)+1} \int_M \beta \wedge *(*d * \alpha) \\
& \overset{(2)}{=} & (-1)^{np+1} \langle \beta, *d * \alpha \rangle.
\end{array}$$

Where (1) follows from the product formula for the differential

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2,$$

for any $\omega_1 \in \Omega^r(M)$ and $\omega_2 \in \Omega(M)$ and (2) follows since $p(p-1)$ is even. $\square$

4.4. **The Hodge Laplacian.** Denote by $\delta = (-1)^{np+1} * d*$ the adjoint[20] of the differential on $\Omega(M)$, in the sense of Proposition 2. Sometimes it will be referred to as the codifferential.

If necessary, we will stress the domain of the differential and the codifferential with an uppercase indicating its source. In other words

$$d^p : \Omega^p(M) \to \Omega^{p+1}(M) \quad , \quad \delta^p : \Omega^p(M) \to \Omega^{p-1}(M).$$

Definition 7. *The (Hodge) Laplacian is the operator*

$$\Delta^p : \Omega^p(M) \to \Omega^p(M)$$

given by $\Delta^p = d^{p-1}\delta^p + \delta^{p+1}d^p$, for each $0 \leq p \leq n$.

It follows from Proposition 2 that Δ is self-adjoint.

Corollary 2. $\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$.

Proof. Formally

$$\begin{array}{rcl}
\langle \Delta\alpha, \beta \rangle & = & \langle (d\delta + \delta d)\alpha, \beta \rangle \\
& = & \langle \delta\alpha, \delta\beta \rangle + \langle d\alpha, d\beta \rangle \\
& = & \langle \alpha, (d\delta + \delta d)\beta \rangle \\
& = & \langle \alpha, \Delta\beta \rangle.
\end{array}$$

It is a simple exercise to make it precise. $\square$

[20] Observe that it is not an adjoint at each $\Omega^p(M)$, since $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$.

It is also easy to see that $*\Delta = \Delta*$ (Exercise 6.1 in Warner), though here we need to be a little more careful (because of sign issues). Namely

$$\begin{aligned} *\Delta^p &= *d^{p-1}\delta^p + *\delta^{p+1}d^p \\ &= (-1)^{n(p-1)+1} * d^{p-1} * d^{n-p} * + (-1)^{np+1} * * d^{n-p-1} * d^p \\ &= (-1)^{n(p-1)+1} * d^{p-1} * d^{n-p} * + (-1)^{np+1+(n-p)p} d^{n-p-1} * d^p \\ &= (-1)^{n(p-1)+1} * d^{p-1} * d^{n-p} * + (-1)^{1-p^2} d^{n-p-1} * d^p, \end{aligned}$$

and similarly

$$\begin{aligned} \Delta^{n-p}* &= d^{n-p-1}\delta^{n-p} * + \delta^{n-p+1} d^{n-p} * \\ &= (-1)^{n(n-p-1)+1} d^{n-p-1} * d^p * * + (-1)^{n(n-p)+1} * d^{p-1} * d^{n-p} * \\ &= (-1)^{n(n-p-1)+1+p(n-p)} d^{n-p-1} * d^p + (-1)^{n(n-p)+1} * d^{p-1} * d^{n-p} * \\ &= (-1)^{1-p^2} d^{n-p-1} * d^p + (-1)^{n(n-p)+1} * d^{p-1} * d^{n-p} * . \end{aligned}$$

Noting that $n(n-p) + 1 + n(p-1) + 1 = n^2 - n + 2$ is even, we see that $*\Delta^p = \Delta^{n-p}*$.

Proposition 3. $\Delta\alpha = 0$ *if and only if* $d\alpha = 0$ *and* $\delta\alpha = 0$.

Proof. The "only if" direction is evident. To see the "if" part, we recycle some previous calculations to see that

$$\begin{aligned} 0 &= \langle \Delta\alpha, \alpha \rangle \\ &= \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle \\ &= \|d\alpha\|^2 + \|\delta\alpha\|^2. \end{aligned}$$

It follows that $\|d\alpha\| = \|\delta\alpha\| = 0$, and we are done. □

Corollary 3 (Maximum Principle)**.** *The only harmonic functions* ($\Delta f = 0$) *on a compact, connected and oriented Riemannian manifold are the constant functions.*

5. The Hodge theorem and consequences

We will follow the presentation in Warner. The advantage of this proof is that the decomposition theorem[21] can be stated and proved very soon, assuming a couple of difficult results. Proving those two results will demand a big deal of analysis and most of the time of these lectures. In order not to forget why are we doing this, immediately after proving Hodge's decomposition we will see some of its corollaries.

5.1. **Trying to solve "Poisson's equation".** Let Δ^* be the adjoint of the Laplacian[22] on $\Omega^p(M)$. Note that if

$$\Delta\omega = \alpha \tag{6}$$

then for any $\varphi \in \Omega^p(M)$ we have $\langle \Delta\omega, \varphi\rangle = \langle \alpha, \varphi\rangle$, from which we obtain the identity

$$\langle \omega, \Delta^*\varphi\rangle = \langle \alpha, \varphi\rangle,$$

for all $\varphi \in \Omega^p(M)$. Since each $\omega \in \Omega^p(M)$ determines a bounded linear functional, given by

$$\ell(\beta) = \langle \omega, \beta\rangle, \tag{7}$$

we see that if equation (6) holds, then ℓ satisfies the equation

$$\ell(\Delta^*\varphi) = \langle \alpha, \varphi\rangle, \tag{8}$$

for all $\varphi \in \Omega^p(M)$.

The main point of this rather trivial observations is that we do not know who ω is, we can try to ask better who ℓ is. In this case, we say that the bounded linear functional $\ell : \Omega^p(M) \to \mathbb{R}$ satisfying equation (8) is a weak solution of equation (6). Unbelievably enough, this turns out to be really useful and many times easier[23].

The obvious question now is: "Imagine we find a weak solution…how do we know a (classical, strong) solution of $\Delta\omega = \alpha$ exists?" This is the hardest part of Hodge theory[24] and what will consume most of our time. Note that what we need to prove is that, provided we have a weak solution ℓ, then there is a smooth form $\omega \in \Omega^p(M)$ *representing* ℓ, in other words, that ℓ can be written as in (7). Once we have that, our problem is solved, since

$$\langle \Delta\omega, \varphi\rangle = \langle \omega, \Delta^*\varphi\rangle = \ell(\Delta^*\varphi) = \langle \alpha, \varphi\rangle$$

holds for every $\varphi \in \Omega^p(M)$.

[21]Whose statement was presented in the Subsection 1.1.

[22]Which is Δ itself.

[23]As a matter of fact, the finite element method for numerical solutions of PDEs is based in this very principle. What people look for are convenient weak solutions of the original PDE.

[24]And, according to Atiyah's account, the point where Hodge himself made several mistakes and analysts had to fix them.

5.2. **Regularity theorems and Hodge's decomposition.** Let us just state Theorems 4 and 5 and believe they are true for a while.

Theorem 4. *Let $\alpha \in \Omega^p(M)$ and let ℓ be a weak solution of* (6)*. Then there exists $\omega \in \Omega^p(M)$ such that* (7) *holds for every $\beta \in \Omega^p(M)$.*

Theorem 5. *Let $\{\alpha_n\}$ be a sequence in $\Omega^p(M)$ such that $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for some $c > 0$. Then there is a Cauchy subsequence of $\{\alpha_n\}$ in $\Omega^p(M)$.*

The proof of both need many tools from analysis and our full attention, but in order not to miss the main point of the exposition, let us see how does Hodge's decomposition follow from them.

Definition 8. *The space of harmonic $p-$forms, denoted by H^p, is given by*

$$H^p = \{\omega \in \Omega^p(M) : \Delta\omega = 0\}.$$

In the proof of Theorem 6 we will need the following Lemma. This will illustrate a more or less typical application of Theorems 4 and 5.

Lemma 2. *Let $(H^p)^\perp$ be the orthogonal complement to H^p. There is a constant $c > 0$ such that*

$$\|\beta\| \leq c\|\Delta\beta\|,$$

for all $\beta \in (H^p)^\perp$.

Proof. Suppose there is a sequence β_j in $(H^p)^\perp$ with $\|\beta_j\| = 1$ and $\|\Delta\beta_j\| \to 0$. By Theorem 5 we can assume $\{\beta_j\}$ to be a Cauchy sequence. This implies the following functional

$$\ell(\psi) = \lim_{j\to\infty} \langle \beta_j, \psi \rangle$$

is well defined (i.e. the limit exists) for all $\psi \in \Omega^p(M)$. It is easily seen that ℓ is bounded and moreover

$$\ell(\Delta\varphi) = \lim_{j\to\infty} \langle \beta_j, \Delta\varphi \rangle = \lim_{j\to\infty} \langle \Delta\beta_j, \varphi \rangle = 0,$$

So ℓ is a weak solution of $\Delta\beta = 0$. By Theorem 4 there is a $\beta \in \Omega^p(M)$ such that $\ell(\psi) = \langle \beta, \psi \rangle$. Clearly $\beta_j \to \beta$. Since $\|\beta_j\| = 1$ and $\beta_j \in (H^p)^\perp$, then $\|\beta\| = 1$ and $\beta \in (H^p)^\perp$. Again by Theorem 4 we have that $\Delta\beta = 0$, which is impossible. □

And now… la pièce de résistance.

Theorem 6 (Hodge's Decomposition)**.** *For each integer $1 \leq p \leq n$, the space of harmonic $p-$forms H^p is finite dimensional and we have the orthogonal decompositions*

$$\begin{aligned} \Omega^p(M) &= \Delta(\Omega^p(M)) \oplus H^p \\ &= d\delta(\Omega^p(M)) \oplus \delta d(\Omega^p(M)) \oplus H^p \\ &= d(E^{p-1}(M)) \oplus \delta(E^{p+1}) \oplus H^p. \end{aligned} \tag{9}$$

We can see easily that we have the following Corollary.

Corollary 4. *The equation $\Delta\omega = \alpha$ has a solution $\omega \in \Omega^p(M)$ if and only if α is orthogonal to H^p.*

Now we come back to the proof of Theorem 6.

Proof of Theorem 6. The finite dimensionality of H^p follows easily from Theorem 5. Namely, if $\{\alpha_n\}$ is an infinite orthonormal sequence in H^p

$$\|\alpha_n\| = 1 \le 1 \quad \text{and} \quad \|\Delta\alpha_n\| = \|0\| = 0 \le 1,$$

then it should have a Cauchy subsequence, which is of course impossible.

Observe that the discussions in Subsection 4.4 imply that we only need to prove the first line in (9).

Let $\omega_1, \ldots, \omega_s$ be an orthonormal basis of H^p, then any $\alpha \in \Omega^p(M)$ can be written uniquely as

$$\alpha = \beta + \sum_{i=1}^{s} \langle \alpha, \omega_i \rangle \omega_i,$$

where $\beta \in (H^p)^\perp$. From this it follows that we have an orthogonal decomposition

$$\Omega^p(M) = (H^p)^\perp \oplus H^p.$$

We need to prove that $(H^p)^\perp = \Delta(\Omega^p(M))$. Denote by $H : \Omega^p(M) \to H^p$ the corresponding projection operator.

$\Delta(\Omega^p(M)) \subset (H^p)^\perp$: If $\omega \in \Omega^p(M)$ and $\alpha \in H^p$, then

$$\langle \Delta\omega, \alpha \rangle = \langle \omega, \Delta\alpha \rangle = 0.$$

In other words, every element in the image of Δ is orthogonal to any harmonic form.

$(H^p)^\perp \subset \Delta(\Omega^p(M))$: Let $\alpha \in (H^p)^\perp$ and consider the linear functional $\ell : \Delta(\Omega^p(M)) \to \mathbb{R}$ defined by

$$\ell(\Delta\varphi) = \langle \alpha, \varphi \rangle,$$

where $\varphi \in \Omega^p(M)$. It is not hard to believe that ℓ is well defined. But just to be sure, let us notice that if $\Delta\varphi_1 = \Delta\varphi_2$, then $\varphi_1 - \varphi_2 \in H^p$, and so $\langle \alpha, \varphi_1 - \varphi_2 \rangle = 0$. We claim that ℓ is a bounded functional on $\Delta(\Omega^p(M))$, for if $\varphi \in \Omega^p(M)$ and $\psi = \varphi - H(\varphi)$, then

$$\begin{aligned} |\ell(\Delta\varphi)| &= |\ell(\Delta\psi)| = |\langle \alpha, \psi \rangle| \\ &\le \|\alpha\|\|\psi\| \\ &\overset{\text{Lemma 2}}{\le} c\|\alpha\|\|\Delta\psi\| = c\|\alpha\|\|\Delta\varphi\|. \end{aligned}$$

By Hahn-Banach, ℓ extends to a bounded linear functional on $\Omega^p(M)$. Thus ℓ is a weak solution of $\Delta\omega = \alpha$. By Theorem 4, there is $\omega \in \Omega^p(M)$ such that $\Delta\omega = \alpha$, and thus $\alpha \in (H^p)^\perp$ is in the image of Δ.

$\square$

5.3. **Green's operator and cool corollaries.** As promised, we are about to see some of the very nice corollaries of Hodge's decomposition theorem. In order to do this, we need to introduce yet another concept.

Definition 9. *The Green's operator $G : \Omega^p(M) \to (H^p)^\perp$ assigns to each $\alpha \in \Omega^p(M)$ the unique solution of $\Delta\omega\alpha - H(\alpha)$ in $(H^p)^\perp$.*

Proposition 4. *G is bounded, self-adjoint and takes bounded sequences into sequences with Cauchy subsequences.*

Proof. I'll do it later. Promise :).

Anyway, the trick is to see that

$$G = (\Delta|_{(H^p)^\perp})^{-1} \circ \pi_{(H^p)^\perp},$$

where $\pi_{(H^p)^\perp} : \Omega^p(M) \to (H^p)^\perp$ is the obvious projection. $\square$

With this in mind, we can prove the following

Proposition 5. *G commutes with any linear operator that commutes with the Laplacian.*

Proof. Suppose $T\Delta = \Delta T$, and $T : \Omega^p(M) \to \Omega^q(M)$. By commutativity, $T(H^p) \subset H^q$ and since $(H^p)^\perp = \Delta(\Omega^p(M))$, we also have $T((H^p)^\perp) \subset (H^q)^\perp$. This implies

$$T \circ \pi_{(H^p)^\perp} = \pi_{(H^q)^\perp} \circ T,$$

and on $(H^p)^\perp$

$$T \circ (\Delta|_{(H^p)^\perp}) = (\Delta|_{(H^q)^\perp}) \circ T,$$

which implies

$$T \circ (\Delta|_{(H^p)^\perp})^{-1} = (\Delta|_{(H^q)^\perp})^{-1} \circ T.$$

The Proposition follows. $\square$

Theorem 7. *Each de Rham cohomology class on a compact oriented Riemannian manifold contains a unique harmonic representative.*

Proof. I'll write it later. $\square$

Corollary 5. *The de Rham cohomology groups for a compact orientable differentiable manifold are all finite dimensional.*

Proof. A standard partition of unity argument shows that any differentiable manifold can be equipped with a Riemannian metric. Since the spaces of harmonic forms H^p are all finite dimensional, the corollary follows from Theorem 7. $\square$

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